

B_K using HYP-smeared staggered fermions in $N_f = 2 + 1$ unquenched QCDTaegil Bae,¹ Yong-Chull Jang,¹ Chulwoo Jung,^{2,*} Hyung-Jin Kim,¹ Jangho Kim,¹ Jongjeong Kim,^{1,†} Kwangwoo Kim,¹ Weonjong Lee,^{1,‡} Stephen R. Sharpe,^{3,§} and Boram Yoon¹

(SWME Collaboration)

¹*Lattice Gauge Theory Research Center, FPRD, and CTP,**Department of Physics and Astronomy, Seoul National University, Seoul, 151-747, South Korea*²*Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA*³*Physics Department, University of Washington, Seattle, WA 98195-1560, USA*

(Dated: December 22, 2010)

We present results for kaon mixing parameter B_K calculated using HYP-smeared improved staggered fermions on the MILC asqtad lattices. We use three lattice spacings ($a \approx 0.12, 0.09$ and 0.06 fm), ten different valence quark masses ($m \approx m_s/10 - m_s$), and several light sea-quark masses in order to control the continuum and chiral extrapolations. We derive the next-to-leading order staggered chiral perturbation theory (SChPT) results necessary to fit our data, and use these results to do extrapolations based both on SU(2) and SU(3) SChPT. The SU(2) fitting is particularly straightforward because parameters related to taste-breaking and matching errors appear only at next-to-next-to-leading order. We match to the continuum renormalization scheme (NDR) using one-loop perturbation theory. Our final result is from the SU(2) analysis, with the SU(3) result providing a (less accurate) cross check. We find $B_K(\text{NDR}, \mu = 2 \text{ GeV}) = 0.529 \pm 0.009 \pm 0.032$ and $\hat{B}_K = B_K(\text{RGI}) = 0.724 \pm 0.012 \pm 0.043$, where the first error is statistical and the second systematic. The error is dominated by the truncation error in the matching factor. Our results are consistent with those obtained using valence domain-wall fermions on lattices generated with asqtad or domain-wall sea quarks.

PACS numbers: 11.15.Ha, 12.38.Gc, 12.38.Aw

Keywords: lattice QCD, B_K , CP violation**I. INTRODUCTION**

CP violation was first observed, long ago, in neutral kaon mixing. The long-lived eigenstate K_L has a small CP-even impurity, allowing it to decay into two pions. The amount of this impurity is parametrized by ε , whose experimental value is [1].

$$\varepsilon = (2.228 \pm 0.011) \times 10^{-3}. \quad (1)$$

In the standard model, CP violation is induced by box diagrams involving virtual W and Z bosons, which lead to the prediction

$$\varepsilon = \exp(i\phi_\varepsilon) \sqrt{2} \sin(\phi_\varepsilon) C_\varepsilon \text{Im}\lambda_t X \hat{B}_K + \xi \quad (2)$$

where

$$X = \text{Re}\lambda_c[\eta_1 S_0(x_c) - \eta_3 S_3(x_c, x_t)] - \text{Re}\lambda_t \eta_2 S_0(x_t) \quad (3a)$$

$$\lambda_i = V_{is}^* V_{id} \quad (3b)$$

$$x_i = m_i^2/M_W^2 \quad (3c)$$

$$C_\varepsilon = \frac{G_F^2 F_K^2 m_K M_W^2}{6\sqrt{2}\pi^2 \Delta M_K} \quad (3d)$$

$$\xi = \exp(i\phi_\varepsilon) \sin(\phi_\varepsilon) \frac{\text{Im}A_0}{\text{Re}A_0} \quad (3e)$$

For the values of the parameters, see, e.g., Ref. [2]. The key point is that, aside from known kinematic factors and QCD Wilson coefficients, ε is given in terms of elements of the CKM matrix (occurring in the λ_i) and the hadronic matrix element \hat{B}_K . This matrix element has the form

$$\hat{B}_K = C(\mu) B_K(\mu) \quad (4)$$

$$B_K(\mu) = \frac{\sum_\nu \langle \bar{K}_0 | [\bar{s}\gamma_\nu(1-\gamma_5)d] [\bar{s}\gamma_\nu(1-\gamma_5)d] | K_0 \rangle}{\frac{8}{3} \langle \bar{K}_0 | \bar{s}\gamma_0\gamma_5 d | 0 \rangle \langle 0 | \bar{s}\gamma_0\gamma_5 d | K_0 \rangle} \quad (5)$$

where $C(\mu)$ is the Wilson coefficient which makes \hat{B}_K RG (renormalization group) invariant (as discussed further in Sec. III B), μ is the renormalization scale of the operator, and we have assumed that the kaons are at rest. Thus if one can determine the value of \hat{B}_K , one can constrain the CP-violating part of the CKM matrix.

It has been a long-standing goal of lattice QCD to calculate B_K . The challenging part of the required ratio of matrix elements, Eq. (5), is the numerator. This is,

* E-mail at chulwoo@bnl.gov

† Current address: Physics Department, University of Arizona, Tucson, AZ 85721, USA

‡ E-mail at wlee@snu.ac.kr; Home page at <http://lgt.snu.ac.kr/>; Visiting professor at Physics Department, University of Washington, Seattle, WA 98195-1560, USA§ E-mail at sharpe@phys.washington.edu; Home page at <http://www.phys.washington.edu/users/sharpe/>

at first sight, relatively straightforward to evaluate, because it involves only a single, stable particle in both initial and final states. A complication arises, however, from the left-handed structure of the four-fermion operator. This constrains its behavior in and around the chiral limit. In particular, at leading order in chiral perturbation theory, the matrix element in the numerator is proportional to m_K^2 . This vanishing is not guaranteed if one uses a formulation of lattice fermions in which chiral symmetry is broken. Because of this, most recent lattice calculations of B_K have used fermions in which one has at least some remnant of chiral symmetry—domain-wall, staggered, overlap and twisted-mass fermions.

After a long history of quenched calculations, advances in algorithms, methodology and computers have allowed calculations of B_K using light dynamical quarks. Particularly noteworthy are two “2 + 1 flavor” calculations in which all three of the physical light quarks are dynamical, although the up and down quarks are kept degenerate.¹ These calculations both use valence domain-wall fermions, with one using asqtad staggered sea quarks [3], and the other domain-wall sea quarks [4, 5]. Both calculations have produced results in which all sources of systematic error are controlled, and the total error is less than 5%. In particular, both calculations use two lattice spacings to allow a continuum extrapolation. The results of these two calculations are consistent.

It is important for a parameter such as B_K , which is used to constrain the standard model, to have multiple calculations using different lattice methods. In particular, we think it is important to use different valence fermions. Thus we have undertaken a calculation using valence staggered fermions. In particular, we use HYP-smeared improved staggered fermions, since these are known to reduce taste-breaking, and are computationally inexpensive to implement. We use the same asqtad lattices as in Ref. [3], except that we add a third, smaller lattice spacing ($a \approx 0.12, 0.09$ and 0.06 fm). An important feature of our approach is that the four-fermion operators are constructed using HYP-smeared (rather than thin) links. This is known to reduce the size of perturbative corrections [6] and scaling violations [7].

There has been one previous calculation of B_K using valence staggered fermions on the asqtad, $N_f = 2 + 1$ lattices [8]. This work differs from ours in that they used thin links to construct the operators, and only worked on the $a \approx 0.12$ fm lattices. As a consequence, the result contains potentially large discretization and matching errors.

A drawback of using staggered fermions is that each flavor comes with 4 tastes, and one needs to use the rooting prescription to reduce the number of sea quarks to the physical complement. Rooting leads to unphysical effects at non-zero lattice spacing. We assume, following

the arguments of Refs. [9–12], that the effects of rooting vanish in the continuum limit, and are taken into account at non-zero lattice spacing by our use of extrapolations based on staggered chiral perturbation theory (SChPT). The phenomenological successes of rooted staggered fermions [13] provide support to this assumption.

This paper is organized as follows. In Section II, we derive the necessary next-to-leading order (NLO) SChPT results, first for SU(3) ChPT including the effects of our use of a mixed action, and then for SU(2) ChPT. In Section III, we explain our methodology, describe how we obtain results for B_K , and then present our fits using both SU(3) and SU(2) SChPT. In the SU(3) case we must use constrained, Bayesian fitting, given the complexity of the functional form. We also discuss the light sea-quark mass dependence and the continuum extrapolation. We close the section by presenting final results and error budgets for both SU(3) and SU(2) analyses. We conclude in Section IV with a comparison of our result with previous work and some comments on our future plans. In four appendices we present, respectively, the functional forms needed for SU(3) fitting, the functional forms needed for SU(2) fitting, a derivation of several key results needed to obtain the SU(2) functional forms, and tables of a subset of our numerical results.

Preliminary results from this project have been presented in Refs. [14–17].

II. STAGGERED CHIRAL PERTURBATION THEORY

To extrapolate our data to the physical (average) light quark mass, and to the continuum limit, we use staggered chiral perturbation theory (SChPT). We have carried out fits using the functions predicted by both SU(3) (strange quark treated as light) and SU(2) (strange quark treated as “heavy”) SChPT. The purpose of this section, and the associated appendices, is to derive these functions and present them in an explicit form suitable for fitting.

For SU(3), a next-to-leading order (NLO) SChPT analysis of B_K has been given in Ref. [18]. That work, however, considered the theory in which the sea and valence quarks had the same action. In our set-up, by contrast, we have asqtad sea quarks (since we use configurations generated by the MILC collaboration [19]) and HYP-smeared valence quarks (as described in the following section). Thus we must generalize the results of Ref. [18] to a mixed-action theory. We also present the results in an explicit notation rather than using the compact but dense notation of Ref. [18].

The use of SU(2) ChPT has been advocated in Ref. [20], and the continuum partially quenched NLO predictions for a variety of quantities, including B_K , have been presented. These results are directly applicable to domain-wall fermions, and have been used for extrapolating B_K in Ref. [21]. Here we generalize the results to

¹ We also compare to results from calculations with two degenerate dynamical quarks in the concluding section.

staggered ChPT, finding that the complications of the SU(3) NLO SChPT form are greatly reduced.

Before presenting the derivations we describe our notation for the quantities that appear in the SChPT expressions, and explain how we obtain the numerical values for these quantities. We work throughout in the isospin-symmetric limit, and label the up and down sea quarks collectively as l . Thus m_l is the common light sea-quark mass, while m_s is the mass of the strange sea quark. Following Ref. [18], valence down (anti-)quarks are labeled x while valence strange (anti-)quarks are labeled y . These have masses m_x and m_y , respectively. As noted above, the sea quarks use the asqtad action, while the valence quarks use the HYP-smeared staggered action (more precisely the HYP (II) choice in the convention of Refs. [6, 22]).

Many pseudo-Goldstone bosons (PGB) appear in the NLO chiral expressions. The mass of the “kaon” composed of x -quarks and y -antiquarks (or vice-versa) is denoted $m_{xy:B}$, where B is the taste. The taste can be I, P, V, A or T, standing, respectively for taste scalar [1], pseudo-scalar [ξ_5], vector [ξ_μ], axial [$\xi_{\mu 5}$] and tensor [$\xi_{\mu\nu}$]. Here we use the result that at leading order (LO) in SChPT there is an accidental SO(4) taste symmetry [23]. Similarly, the masses of “pions” composed of x -quarks and x -antiquarks are labeled $m_{xx:B}$, while those of PGBs composed of y -quarks and antiquarks are denoted $m_{yy:B}$. In the following, PGB masses are always in physical units, converted from the measured lattice masses using the scales determined by the MILC collaboration [13, 24]. These scales are given in Table IV below.

To simplify the chiral expressions, we also use the following shorthand notations:

- The squared masses of the valence kaons are denoted by $K_B \equiv m_{xy:B}^2$. We also use a special shorthand $G \equiv K_P$ for the Goldstone-taste kaon. This is the particle which we use for the external states in our calculation.
- The squared masses of the valence pions are denoted: $X_B = m_{xx:B}^2$. Similarly, we use $Y_B = m_{yy:B}^2$. Note that for tastes B = I, V and A the pion masses have LO contributions from quark-disconnected diagrams involving “hairpin” vertices. These are, by definition, *not included* in X_B and Y_B . In other words, these mass-squareds are those of the pions in which the quark and antiquark are taken (implicitly) to have different flavors.
- We also need the squared masses of analogous flavor non-singlet pions composed of $\bar{l}l$ and $\bar{s}s$ quarks. We call these $L_B \equiv m_{\bar{l}l:B}^2$ and $S_B \equiv m_{\bar{s}s:B}^2$, respectively, for B = I, V and A.
- Finally, we need the squared masses of the η_B and η'_B pions for B = V, A and I. We call these masses simply η_B and η'_B . The η_B and η'_B are the flavor singlet, sea-quark mesons that result from the in-

clusion of mixing between $\bar{l}l$ and $\bar{s}s$ pions due to hairpin diagrams [25].

We now address how we obtain the numerical values of the various masses discussed above. The valence kaon and non-singlet pion masses K_B , X_B , and Y_B are obtained from our simulations in an ancillary calculation. In fact, we find to an extremely good approximation that, for our range of valence quark masses, the expected LO forms hold:²

$$K_B = b_1(m_x + m_y) + \Delta_B \quad (6a)$$

$$X_B = b_1(2m_x) + \Delta_B \quad (6b)$$

$$Y_B = b_1(2m_y) + \Delta_B. \quad (6c)$$

In practice we do a fit to these equations and extract b_1 and the Δ_B , and then use the latter to reconstruct the mass-squareds. More details on these fits are given below in Sec. III A.

TABLE I. Masses of flavor non-singlet, taste P sea-quark masses, in lattice units [19, 24]. Ensemble labels are from Table III below.

Ensemble	$a\sqrt{L_P}$	$a\sqrt{S_P}$
C1	0.3779	0.4875
C2	0.3113	0.4889
C3	0.2245	0.4944
C4	0.1889	0.4923
C5	0.1597	0.4913
F1	0.1479	0.3273
F2	0.1046	0.3273
S1	0.09371	0.2075

TABLE II. Taste splittings, $r_1^2\Delta_B$, for flavor non-singlet pions composed of sea quarks [24]. These are for the ensembles C3, F1 and S1 (see Table III below).

taste	coarse	fine	superfine
$r_1^2\Delta_A$	0.2053	0.0706	0.0253
$r_1^2\Delta_T$	0.3269	0.1154	0.0413
$r_1^2\Delta_V$	0.4391	0.1524	0.0552
$r_1^2\Delta_S$	0.5370	0.2062	0.0676

In the case of pions composed of sea quarks, we obtain the flavor non-singlet masses L_B and S_B from MILC collaboration results [19, 24]. The sea pion and “ss” meson masses are quoted in Table I, while taste splittings are collected in Table II. We do not include the errors on these splittings in our analysis as their impact is much smaller than other sources of error. This is because the

² SChPT predicts that $\Delta_P = 0$, although in practice the missing NLO terms together with finite-volume effects can lead to a small non-zero value for Δ_P .

errors are very small, and furthermore enter into a numerically small part of the one-loop correction. We use the same taste-splittings for all the coarse ensembles, and similarly for both the fine ensembles, which we expect to be a very good approximation based on our results for valence pions.

For the flavor singlets sea-quark mesons, the taste-singlet is simplest, since the η'_I (which corresponds to the η' of QCD) is not a PGB and can be integrated out. This leads to the following result for the η_I mass: [25]:

$$\eta_I = \frac{1}{3}L_I + \frac{2}{3}S_I. \quad (7)$$

For tastes B=V and A, we use the result of Ref. [25]:

$$\eta_B = m_{\eta:B}^2 = \frac{1}{2} \left(L_B + S_B + \frac{3}{4}a^2\delta_B'^{ss} - Z_B \right), \quad (8)$$

$$\eta'_B = m_{\eta':B}^2 = \frac{1}{2} \left(L_B + S_B + \frac{3}{4}a^2\delta_B'^{ss} + Z_B \right), \quad (9)$$

$$Z_B = \sqrt{H_B^2 - \frac{1}{2}a^2\delta_B'^{ss}H_B + \frac{9}{16}(a^2\delta_B'^{ss})^2}, \quad (10)$$

$$H_B = S_B - L_B. \quad (11)$$

The superscript on the hairpin vertices $a^2\delta_{A,V}'^{ss}$ will be explained in the following section. Their numerical values are taken from the fits of Ref. [19]. Note that their dimensions are given by $\delta_B'^{ss} \propto \Lambda_{\text{QCD}}^4$.

The taste violating parameters on the coarse lattices are [24]

$$r_1^2 a^2 \delta_A' = -0.30(1)(3) \quad (12)$$

$$r_1^2 a^2 \delta_V' = -0.05(2)(3). \quad (13)$$

These come from fits using the “smoothed” r_1 values that we adopt, and that are given in Table IV below. As for the sea-quark taste splittings, we fix these parameters to their central values and ignore the errors. For the other lattice spacings we scale the parameters assuming an $a^2\alpha_s^2$ dependence.

A. Mixed action effects in SU(3) SChPT

The inclusion of mixed-action effects into SChPT can be accomplished by a straightforward generalization of the methods worked out in Refs. [26–29]. A mixed action can be described field-theoretically at the lattice level in the same way as a partially quenched (PQ) theory: there are valence quarks, corresponding ghosts, and sea quarks. The difference is that there is no longer a symmetry interchanging valence and sea quarks. This symmetry only emerges in the continuum limit, where the choice of discretization does not matter. What this means is that one has a standard PQ set up when $a \rightarrow 0$, but, for $a \neq 0$, the corrections introduced by discretization errors in the valence, sea and mixed valence-sea sectors [parametrized by low-energy coefficients (LECs)] are unrelated. At first sight (and in light of the complexity of the analysis of B_K

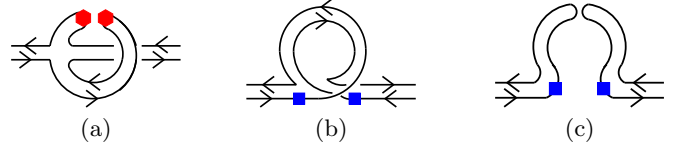


FIG. 1. Classes of quark-line diagrams contributing to B_K at 1-loop order. The four-quark operator is represented as two bilinears which change a valence d -quark into a valence s -quark. These bilinears are shown as either (blue) squares (if the bilinear connects to both quark and antiquark from an external kaon) or as (red) hexagons (if the bilinear connects a quark from one kaon to an antiquark from the other). For each of the three diagrams there are others (not shown) related by interchanging the roles of quark and antiquark. In (c) the “hairpin vertex” contains, implicitly, a sum of “bubbles” composed of sea quarks.

in an unmixed context [18]), the required generalization appears rather daunting. It turns out, however, that this is not the case for our particular application, since nearly all of the new LECs do not enter at NLO in B_K .

The easiest way to see this is to look at the quark-flow diagrams which contribute. These have been presented in Ref. [18], and we display them in Fig. 1. Only valence-quark lines occur in Figs. 1(a) and (b), so that the PGBs entering in loop diagrams are all of the “valence-valence” type. The *form* of the contributions from these diagrams will thus be unchanged by the use of a mixed action, and we can simply use the results of Ref. [18]. The contributions of these diagrams give what is called the “connected” part of the matrix element, $\mathcal{M}_{\text{conn}}$, where connected means that the loops do not involve hairpin vertices. We present a fully explicit form for $\mathcal{M}_{\text{conn}}$ in the next section. We stress that the *values* of the LECs in $\mathcal{M}_{\text{conn}}$ are altered by the change from unmixed to mixed action, but this has no practical consequences as these are *a priori* unknown parameters that are either determined from the spectrum or by the fits to B_K itself.

Another feature of the quark-line diagrams is that no “valence-sea” mesons appear at NLO. Thus the new LECs associated with such mixed mesons do not enter into our result.

New LECs do appear in the “disconnected” matrix element, $\mathcal{M}_{\text{disc}}$, for which the quark-line diagram is shown in Fig. 1(c). The hairpin vertices in the loop implicitly contain a sum over any number of insertions of sea-quark loops, corresponding to the propagation of “sea-sea” PGBs. This means that three types of hairpin vertex enter, instead of the single type for an unmixed action: $a^2\delta_B'^{vv}$, for the hairpin connecting valence-valence pions to themselves; $a^2\delta_B'^{vs}$, for the hairpin connecting valence-valence pions to sea-sea pions, and $a^2\delta_B'^{ss}$, for the hairpin connecting sea-sea pions to themselves. The last of these has already appeared in Eqs. (8-10). These hairpins occur as LO vertices only for tastes B=V and A, so that, for B_K , the change to a mixed action introduces only four new LECs: $\delta_V'^{vv}$, $\delta_A'^{vv}$, $\delta_V'^{vs}$ and $\delta_A'^{vs}$. We stress that these hairpin vertices are independent of the quark masses.

Hairpins also occur at LO in the taste singlet channel, but here the dominant contribution is that which leads to the bulk of the physical η' mass. This is a continuum contribution, which is therefore independent of the fermion action, and thus is common to vv , vs and ss vertices. Small differences between these vertices of $O(a^2)$ do not contribute to B_K until next-to-next-to-leading order (NNLO). Thus the three types of vertices are the same at LO in the taste-singlet channel, and the result of Ref. [18] for this channel is unchanged.

In light of the preceding discussion, we see that the ingredients that change when moving from a PQ (unmixed action) theory to a mixed-action (MA) set-up are the hairpin propagators, $D_{xx}^{B,PQ}(q)$, $D_{xy}^{B,PQ}(q)$, and $D_{yy}^{B,PQ}(q)$, where $B=V$ or A labels the taste. We first recall the form of these propagators in a theory with an unmixed action [18]. The “ xy ” hairpin propagator is

$$D_{xy}^{B,PQ}(q) = -a^2 \delta'_B \frac{1}{(q^2 + X_B)(q^2 + Y_B)} R_B(q), \quad (14)$$

$$R_B(q) = \frac{(q^2 + L_B)(q^2 + S_B)}{(q^2 + \eta_B)(q^2 + \eta'_B)}, \quad (15)$$

[this is Eq. (42) of Ref. [18] restricted to the case of $2+1$ flavors]. The expressions for η_B and η'_B are those given in (8) and (9) above except that δ'^{ss}_B is replaced by the single, common, hairpin vertex δ'_B . The results for $D_{xx}^{B,PQ}$ are simply obtained by the replacement $Y_B \rightarrow X_B$, while the opposite replacement yields the $D_{yy}^{B,PQ}$. Similar replacements hold for the mixed-action theory, and so we discuss only D_{xy} below.

The result (14) can be understood as follows: $a^2 \delta'_B$ is the hairpin vertex, $R_B(q)$ is the result of the sum over sea-sea mesons “within” the hairpin vertex, and the remaining two propagators are those of the “external” xx and yy pions. One can then see how the result changes when the hairpin vertex splits into three types. If there are no sea-quark loops, one has the vv hairpin vertex and the external propagators, but no factor of $R_B(q)$. If there are sea-quark loops, then one has a common factor of $(\delta'^{vs}_B)^2$ (from $vv \rightarrow ss$ and $ss \rightarrow vv$), and the sum $R_B(q)$ except for the first term, with one factor of δ'^{ss}_B removed to avoid double-counting. Thus one finds that $D_{xy}^{B,PQ}(q)$ should be replaced with the mixed-action hairpin propagator

$$D_{xy}^{B,MA}(q) = -\frac{1}{(q^2 + X_V)(q^2 + Y_V)} \left[a^2 \delta'^{vv}_B + (a^2 \delta'^{vs}_B)^2 \frac{(R_B(q) - 1)}{a^2 \delta'^{ss}_B} \right] \quad (16)$$

$$= \left(\frac{(\delta'^{vs}_B)^2}{\delta'^{ss}_B \delta'_B} \right) D_{xy}^{B,PQ}(q) - a^2 \left(\frac{\delta'^{vv}_B \delta'^{ss}_B - (\delta'^{vs}_B)^2}{\delta'^{ss}_B} \right) \frac{1}{(q^2 + X_B)(q^2 + Y_B)}. \quad (17)$$

From the second line, we see that the impact of using a mixed action is twofold.

1. One must replace the overall factor multiplying the loop contribution calculated in the unmixed case as follows:

$$\delta'_B \longrightarrow \delta_B^{MA1} = \frac{(\delta'^{vs}_B)^2}{\delta'^{ss}_B}, \quad B = V, A. \quad (18)$$

The form of the contribution will be otherwise unchanged, since the q dependence in the first term of (17) is the same as that of $D_{xy}^{B,PQ}(q)$. We also note that our lack of knowledge of the two δ_B^{MA1} does not introduce new fit parameters since these hairpin vertices multiply unknown operator coefficients from the chiral theory [as shown explicitly in Eq. (A16)].

2. In addition, one must evaluate a new loop integral, in which propagator has the same form as for a quenched hairpin, and in which the overall constant is

$$\delta_B^{MA2} = \frac{\delta'^{vv}_B \delta'^{ss}_B - (\delta'^{vs}_B)^2}{\delta'^{ss}_B}, \quad B = V, A. \quad (19)$$

The result is given as $F_B^{(2)}$ in Appendix A 3. This does introduce two new fit parameters.

We close this section with a comment on the expected magnitudes of the four new LECs. The hairpin vertices of SChPT are the result of mapping four-quark operators in the Symanzik effective theory into the chiral theory. The structure of the underlying four-quark operators mirrors that of the chiral operators they produce. For example, the hairpin vertex proportional to δ'^{vs}_B arises from an operator containing a valence-quark bilinear and a sea-quark bilinear, while that proportional to δ'^{vv}_B arises from an operator containing two valence-quark bilinears. The important observation is these four-quark operators also give non-hairpin taste-dependent contributions to the pion masses (viewed as having the gluons exchanged in the t-channel rather than the s-channel). The operators leading to δ'^{vv}_B , δ'^{vs}_B and δ'^{ss}_B lead, respectively, to taste splittings for valence-valence, valence-sea and sea-sea pions. But we know that the valence HYP-smear action reduces taste splittings by a factor of ≈ 3 compared to that for the sea-quark asqtad action [30]. Assuming that this is a universal reduction, applicable to all four-quark operators, we conclude that $\delta'^{vv}_B / \delta'^{ss}_B \approx 1/3$. Since the improvement in taste-splittings arises from a reduction of the coupling of quarks to high-momentum gluons, we expect that it “factorizes”, so that $\delta'^{vs}_B / \delta'^{ss}_B \approx 1/\sqrt{3}$. Given

these assumptions, we are led to the expectations

$$\delta_B^{\text{MA1}} \approx \delta_B^{\prime ss}/3 \quad \text{and} \quad \delta_B^{\text{MA2}} \approx 0. \quad (20)$$

If these expectations are close to correct, then the taste V and A hairpin contributions are significantly reduced by the use of the mixed action: the “old” contributions from $D_{xy}^{\text{B,PQ}}(q)$ are reduced by a factor of ~ 3 , while the “new” δ_B^{MA2} terms are suppressed. We use these expectations to simplify our fitting strategy.

B. SU(3) SChPT result

In this subsection we summarize the NLO result for B_K in SU(3) SChPT—obtained from Ref. [18] with the generalization explained in the previous subsection. The utility of this result depends on the extent to which one can treat the strange quark mass as light compared to Λ_{QCD} . We will attempt to judge this *a posteriori* based on how successful we are in fitting with the SU(3) form.

Discretization errors in B_K arise from the action and from the four-quark operator. If they involve taste-breaking then they are, for both HYP-smeared and asqtad actions, of $O(a^2\alpha^2)$, while non taste-breaking errors are of $O(a^2)$ for HYP-smeared quarks and of $O(a^2\alpha)$ for asqtad quarks. The different factors of α are, however, misleading, since numerical results indicate that taste-breaking effects are enhanced. Furthermore, as noted above, different actions can lead to significantly different numerical sizes of taste-breaking effects, although formally these effects are of the same order. Thus, following Ref. [18], we adopt the phenomenologically-based power-counting scheme in which all discretization errors are treated as being of $O(p^2)$, irrespective of the number of powers of α . In light of this choice, and for the sake of brevity, we refer below to all discretization errors as being simply of “ $O(a^2)$ ”, except where the associated powers of α are important.

Additional errors arise if one uses perturbatively calculated matching factors to relate the lattice operator to its continuum counterpart. We use one-loop matching, and furthermore keep only the subset of the one-loop induced operators involving bilinears with taste P. This results in “mixing errors” proportional to both α_s —from one-loop mixing with operators with tastes other than P—and α_s^2 —from the unknown two-loop mixing with all operators, including those with taste P. As explained in Ref. [18], it is reasonable phenomenologically to treat both these effects as also of $O(p^2)$. The justification for treating $O(\alpha) \sim O(\alpha^2)$ is that the one-loop coefficients of the operators we drop are known to be small, while two-loop coefficients are unknown and might not be small. Indeed, with HYP-smeared fermions, we find that the relevant one-loop coefficients are of typical size $\alpha_s/(4\pi) \approx 0.08\alpha_s$ [22, 31], and thus expect that their effects are noticeably smaller than those of $O(\alpha_s^2)$. In light of this, we refer to truncation errors below as simply of $O(a^2)$ unless we need to be more specific.

Using this power-counting scheme, Ref. [18] determined all the operators in SChPT that must be included at NLO, and carried out the required one-loop calculation. This involves using LO propagators in the loops, which implies that the accidental SO(4) taste symmetry [23, 25] holds for their masses. As already noted, we determine these masses from an ancillary spectrum calculation on the same configurations, or from published MILC results, so that they are not unknowns in the fits.

The resulting expression for B_K can be written as

$$B_K^{\text{SU}(3)} = \sum_{j=1}^{16} b_j H_j. \quad (21)$$

Here the coefficients b_j are related to unknown SChPT LECs, while the H_j are known functions of the (numerically determined) PGB masses. In the following we run through these 16 terms in turn, explaining their origin and their properties. The explicit expressions for the H_j are collected in Appendix A.

The dominant contribution to B_K comes from the function

$$H_1 = 1 + \frac{1}{8\pi^2 f_\pi^2 G} [\mathcal{M}_{\text{conn}}^0 + \mathcal{M}_{\text{disc}}^0]. \quad (22)$$

This contains the LO contribution (the “1”) together with the “continuum-like” loop contributions to $\mathcal{M}_{\text{conn}}$ and $\mathcal{M}_{\text{disc}}$. By “continuum-like” we mean that the logarithms have the continuum form (thus the superscript 0, indicating $a \rightarrow 0$), except that the pions in the loops have taste-splittings included in their masses. If these taste-splittings are set to zero, then $\mathcal{M}_{\text{conn}}^0$ and $\mathcal{M}_{\text{disc}}^0$ go over to the continuum chiral logarithms. Assuming that f_π is known, the coefficient of the chiral logarithms is predicted.³ Since we know the values of the required PGB masses, we know the value of the function H_1 for each choice of valence and sea-quark masses. For this reason it makes sense to lump these terms together into a single function, H_1 . The explicit forms for $\mathcal{M}_{\text{conn}}^0$ and $\mathcal{M}_{\text{disc}}^0$ are given in eqs. (A1) and (A8), respectively. It turns out that $\mathcal{M}_{\text{disc}}^0$ vanishes for degenerate valence quarks.

The chiral logarithms in (22) have the generic form $m_K^2 \ln m_K$. Since an explicit $1/G$ factor is taken out, the functions $\mathcal{M}_{\text{conn}}^0$ and $\mathcal{M}_{\text{disc}}^0$ have generic form $m_K^4 \ln m_K$.

The coefficient of H_1 is given by

$$b_1 = B_0 + O(a^2, \alpha^2). \quad (23)$$

Here B_0 is the value of B_K in the combined $SU(3)$ chiral and continuum limits. B_0 is a quantity that is of interest for continuum ChPT. The remaining terms indicate the manner in which b_1 approaches the continuum limit:

³ As discussed below, our data is insufficient to determine f_π from the fits. Thus we use a fixed value, which we vary among reasonable choices, as discussed below.

there are contributions from both non-taste breaking and taste-breaking discretization errors, and from the truncation of perturbation theory.⁴ When fitting at a single lattice spacing, one can treat b_1 as a single constant. The scaling violations must be accounted for when extrapolating to the continuum limit. Alternatively one can attempt a combined continuum-chiral fit, as has been done by the MILC collaboration [19]. We have not done so here.

The next four terms are the analytic NLO (and, in the case of H_3 , NNLO) corrections:

$$H_2 = G/\Lambda_\chi^2 \quad (24)$$

$$H_3 = (G/\Lambda_\chi^2)^2 \quad (25)$$

$$H_4 = \frac{(X_P - Y_P)^2}{G\Lambda_\chi^2} \quad (26)$$

$$H_5 = \frac{L_P + S_P/2}{\Lambda_\chi^2} \quad (27)$$

All of these terms are present in the continuum limit, and to the order we work their coefficients are independent of a and the quark masses. We have chosen to write them in terms of the PGB masses rather than the quark masses. This allows us to set the scale of these terms by the chiral scale Λ_χ which we take to be 1 GeV in our fitting. Then, assuming naive dimensional analysis, we expect the coefficients of these terms $b_2 - b_4$ to be of $O(1)$.

We find that the NNLO analytic term, $b_3 H_3$, is necessary to obtain reasonable fits. The need for a NNLO term is not surprising given relatively large kaon masses included in our SU(3) fits. It is, perhaps, surprising that we can get away with only a single term beyond NLO, given that the MILC collaboration requires NNNLO and NNNLO terms in their fits to pion and kaon properties [19]. The difference appears to be that the errors in B_K are larger than those in the quantities studied by the MILC collaboration—PGB masses and decay constants.

We stress that we are not including the complete NNLO expression (which is not known). In particular, we are not including non-analytic NNLO terms, and are keeping only one of the NNLO analytic terms. Our approach here is phenomenological—we are keeping the minimum number of NNLO analytic terms needed to obtain reasonable fits.

The remaining 11 H_j are pure lattice artifacts, caused either by discretization errors or the truncation of matching factors, or both. They all arise from taste-violating interactions.⁵ The first 7 are corrections to $\mathcal{M}_{\text{conn}}$, and

are labeled by

$$H_6 = F_1^{(4)} \quad (28)$$

$$H_7 = F_P^{(4)} \quad (29)$$

$$H_8 = F_V^{(4)} \quad (30)$$

$$H_9 = F_A^{(4)} \quad (31)$$

$$H_{10} = F_T^{(4)} \quad (32)$$

$$H_{11} = F^{(5)} \quad (33)$$

$$H_{12} = F^{(6)}. \quad (34)$$

The explicit functional forms of $F_B^{(4)}$, $F^{(5)}$, and $F^{(6)}$ are given in eqs. (A9), (A10) and (A11), respectively. The subscript on $F_B^{(4)}$ indicates the taste of the pion in the loop. The functions $F^{(5)}$ and $F^{(6)}$ vanish in the limit of degenerate valence quarks.

The chiral logarithms in the functions H_{6-12} are enhanced over the continuum-like logarithms in H_1 . While the latter are of standard form $m_K^2 \ln m_K$, the former are simply proportional to $\ln m_K$. This enhancement of the chiral logarithm is balanced by the fact that the coefficient, b_j , is small:

$$b_{6-12} \sim \Lambda_{\text{QCD}}^2 \times (a^2 \Lambda_{\text{QCD}}^2 \text{ or } \alpha^2). \quad (35)$$

The final four functions H_j with $j = 13 - 16$ arise from corrections to $\mathcal{M}_{\text{disc}}$ due to discretization and truncation errors. These are the functions which are affected by our use of a mixed action, as explained above in Sec. II A. H_{13} and H_{14} are the loop integrals involving the hairpin propagator $D_{xy}^{B,\text{PQ}}(q)$ [i.e. the first term in Eq. (17)]. They are, for B=V and A, respectively,

$$H_{13} = F_V^{(1)}, \quad (36)$$

$$H_{14} = F_A^{(1)}, \quad (37)$$

and are multiplied by coefficients proportional to δ_V^{MA1} and δ_A^{MA1} . This leaves the two functions multiplied by the two additional LECs introduced by our use of a mixed action, i.e. δ_V^{MA2} and δ_A^{MA2} . These are, respectively,

$$H_{15} = F_V^{(2)}, \quad (38)$$

$$H_{16} = F_A^{(2)}. \quad (39)$$

The functions $F_B^{(1)}$ and $F_B^{(2)}$ are given in Appendix (A3). They are yet more divergent in the chiral limit than $F^{(4-6)}$, behaving generically as $\ln m_K/m_K^2$ (although they vanish when $m_x = m_y$). The power-counting is restored by coefficients [whose explicit expressions in terms of chiral coefficients are given in eqs. (A16) and (A18)] which are proportional to the product of two factors arising from discretization or truncation errors:

$$b_{13-16} \sim \Lambda_{\text{QCD}}^2 \times (a^2 \delta_B^{\text{MA1}} \text{ or } a^2 \delta_B^{\text{MA2}}) \times (a^2 \Lambda_{\text{QCD}}^2 \text{ or } \alpha^2). \quad (40)$$

⁴ In this case there are no terms linear in α [18].

⁵ There are also taste-conserving discretization and truncation errors; these lead to the corrections in the expression for the coefficient b_1 , Eq. (23).

(We recall that $\delta_B^{\text{MA}1,2} \propto \Lambda_{\text{QCD}}^4$.) As discussed at the end of Sec. II A, we expect $\delta_B^{\text{MA}1}$ to be suppressed by a factor of 3 from its natural size, and for $\delta_B^{\text{MA}2}$ to be close to zero. Thus we expect the contributions from H_{13} and H_{14} to be small, while those from H_{15} and H_{16} to be negligible.

C. SU(2) SChPT result

In the recent literature there has been much discussion of the convergence and reliability of SU(3) ChPT when extrapolating lattice results to the physical kaon mass. As examples, we contrast the apparently successful extrapolations of Refs. [3, 13, 19] (the latter being for B_K using a domain-wall valence/staggered sea mixed action), with the lack of convergence found in Ref. [4, 32]. Recent reviews of the situation are given in Refs. [33, 34], and a related discussion in continuum ChPT is in Ref. [35]. Our view is that whether SU(3) ChPT can be used reliably depends on the quantity considered (the rate of convergence is not universal) and on whether one includes NNLO and higher order terms (these are needed to represent data in the region of the physical kaon mass). Thus we have attempted SU(3) fits using the theoretical form described in the previous subsection (which includes one NNLO term), and find the reasonably successful results to be described below.

It is nevertheless true that, for the non-degenerate lattice kaons which lie closest to the physical kaon, and which thus have the dominant effect on the extrapolation, the non-analytic dependence comes dominantly from chiral logarithms involving *only* the light quarks (x and ℓ in our notation). Chiral logarithms of mesons containing strange quarks (i.e. of K_B , S_B , Y_B , η_I and η'_B ⁶) can be expanded about their values when the strange quark mass equals its physical value and represented accurately by analytic terms. This means that, in effect, one is using an (approximate) SU(2) ChPT expression for those kaons closest to the physical kaon even when nominally doing an SU(3) extrapolation.

The RBC collaboration were led, by considerations along these lines, together with the poor convergence they found for SU(3) ChPT, to propose the use of SU(2) ChPT for the extrapolation of kaon (and pion) properties [4]. In this approach, first considered systematically in Ref. [36] and extended in Refs. [4, 21], the kaon is treated as a heavy, static source, and no expansion is made in the mass of the (valence or sea) strange quark. The expansion in powers of $(m_\pi/\Lambda_\chi)^2$ is supplemented by an expansion in $(m_\pi/m_K)^2$. In our calculation, this means that the SU(2) ChPT result will apply only to the

subset of our non-degenerate masses in which m_x is small and m_y is large.

This methodology has been applied successfully to a calculation of B_K using domain-wall fermions by the RBC collaboration [21]. Since they use lattice fermions with an almost exact chiral symmetry, the corresponding ChPT result is that of the continuum, except that the LECs can depend on a^2 . For our calculation, however, what is needed is an SU(2) *staggered* ChPT result. At first sight, this would appear to require a generalization of the rather involved enumeration of operators performed in Ref. [18] to the case of SU(2) chiral symmetry with a heavy kaon source. It turns out, however, that a simpler approach suffices. One can show that it is sufficient to consider the $m_x, m_\ell \ll m_y \approx m_s$ limit of the *next-to-leading order* SU(3) SChPT result, provided that one allows the LECs to depend on m_y and m_s in an unknown (but analytic) way. In addition, one can show that the size of the chiral logarithms involving m_x and m_ℓ relative to the LO term is *not corrected by an unknown m_s dependence*. Thus the chiral logarithm remains a predicted correction, as in SU(3) SChPT.

The validity of this “recipe” is demonstrated in Appendix C. The argument holds to all orders in an expansion in powers of m_s/Λ_{QCD} .

Applying this recipe we find an important simplification: the NLO terms in SU(3) SChPT that involve taste-breaking LECs proportional to a^2 or α_s^2 get pushed to NNLO in SU(2) SChPT. This drastically reduces the number of parameters required in a fit to the NLO expression. How this happens is explained in detail in Appendix B.

The final SU(2) SChPT result for B_K is

$$B_K = \sum_{i=1}^4 d_i Q_i, \quad (41)$$

where the functions that appear are

$$Q_1 = 1 + \frac{1}{32\pi^2 f^2} \left[(L_I - X_I) \tilde{\ell}(X_I) + \ell(X_I) - 2 \sum_B \tau^B \ell(X_B) \right], \quad (42)$$

$$Q_2 = \frac{X_P}{\Lambda_\chi^2}, \quad (43)$$

$$Q_3 = \left(\frac{X_P}{\Lambda_\chi^2} \right)^2, \quad (44)$$

$$Q_4 = \frac{L_P}{\Lambda_\chi^2}. \quad (45)$$

The chiral-logarithmic functions ℓ and $\tilde{\ell}$ are defined in Eqs. (A4) and (A5), and the factors τ^B , which are the fractional multiplicities of the different tastes, are given in Eq. (A3). Note that we have kept a single analytic NNLO term, Q_3 , as for the SU(3) SChPT fits. As in the

⁶ We do not include η_V or η_A in this list because the strange-quark component is small and these mesons are light. This can be seen from Eq. (8), which in the limit that $a^2 \delta_B' \ll |S_B - L_B|$, which is the case in practice, gives $\eta_B \approx L_B + a^2 \delta_B'$.

SU(3) fits we fix $\Lambda_\chi = 1 \text{ GeV}$, although we could also use $\Lambda_\chi = m_K^{\text{phys}}$.

The constants d_i are arbitrary, unknown, analytic functions of m_s and m_y . In addition, at NLO d_1 contains taste-conserving discretization and truncation errors (which it inherits from b_1). Specifically, one has

$$d_1 = B_0^{SU(2)}(m_y, m_s) + O(a^2, \alpha^2). \quad (46)$$

where the first term is the value of B_K in the SU(2) chiral limit ($m_x = m_\ell = 0$) and with the given values of m_y and m_s . The $O(a^2, \alpha^2)$ terms also have an implicit dependence on m_y and m_s .

Clearly the SU(2) SchPT result is much simpler than that in SU(3) SchPT—at NLO, it has only 3 unknown coefficients at a fixed lattice spacing, to be compared to 16 for the SU(3) form. This gain is compensated in part by the fact that it can be used only for a small subset of our data, and because the fit parameters have an implicit dependence on m_y and m_s . Nevertheless, we find, as described below, that the SU(2) fitting is more straightforward and leads to smaller final errors in B_K .

III. DATA ANALYSIS

In this section we explain how we calculate B_K , how we convert the numerical data into physical observables using matching factors, and how we extrapolate to the physical kaon mass and the continuum limit using SChPT.

A. Computation of B_K

TABLE III. MILC lattices used for the numerical study. Here, “ens” represents the number of gauge configurations, “meas” is the number of measurements per configuration, and ID will be used later to identify the corresponding MILC lattice.

a (fm)	am_l/am_s	size	ens \times meas	ID
0.12	0.03/0.05	$20^3 \times 64$	564×1	C1
0.12	0.02/0.05	$20^3 \times 64$	486×1	C2
0.12	0.01/0.05	$20^3 \times 64$	671×9	C3
0.12	0.01/0.05	$28^3 \times 64$	274×8	C3-2
0.12	0.007/0.05	$20^3 \times 64$	651×10	C4
0.12	0.005/0.05	$24^3 \times 64$	509×1	C5
0.09	0.0062/0.031	$28^3 \times 96$	995×1	F1
0.09	0.0031/0.031	$40^3 \times 96$	678×1	F2
0.06	0.0036/0.018	$48^3 \times 144$	744×2	S1

We use the MILC ensembles listed in Table III. They are generated with $N_f = 2+1$ flavors of improved (asqtad action) staggered sea quarks, using the rooting trick to cancel the effects of the extra tastes.

The values of the light sea-quark mass (am_l) and strange sea-quark mass (am_s) are given in Table III. Details of the lattice generation, decorrelation times, etc.

are given in Ref. [19]. We use three lattice spacings— $a \approx 0.12, 0.09$ and 0.06 fm —which are called coarse, fine and superfine, respectively. These lattice spacings are the nominal values—in our analysis we actually use the values obtained on each ensemble from the value of r_1/a obtained by the MILC collaboration [13, 24]. These are listed in Table IV. To obtain a we set $r_1 = 0.3108(15)_{(-79)}^{(+26)} \text{ fm}$, as given in Ref. [13] using f_π to set the scale.

TABLE IV. Values for r_1/a and $1/a$. See text for discussion.

ID	r_1/a	$1/a$ (GeV)
C1	2.650(4)	1.682(3)
C2	2.644(3)	1.679(2)
C3	2.618(3)	1.662(2)
C3-2	2.618(3)	1.662(2)
C4	2.635(3)	1.673(2)
C5	2.647(3)	1.681(2)
F1	3.699(3)	2.348(2)
F2	3.695(4)	2.346(3)
S1	5.296(7)	3.362(4)

TABLE V. Valence quark masses (in lattice units).

a (fm)	am_x and am_y	
0.12	$0.005 \times n$	with $n = 1, 2, 3, \dots, 10$
0.09	$0.003 \times n$	with $n = 1, 2, 3, \dots, 10$
0.06	$0.0018 \times n$	with $n = 1, 2, 3, \dots, 10$

The valence quarks are HYP-smeared staggered fermions, i.e. the configurations are HYP-smeared [37] (using the coefficients of the HYP(II) choice in the convention of Refs. [6, 22]) and then we use the unimproved staggered action. We have found this to be as effective at reducing taste-breaking as the more elaborate HISQ action [30]. We use 10 different values of the valence quark masses (*i.e.* m_x and m_y) running from $\sim m_s/10$ to $\sim m_s$. They are given in Table V. In total, we have 55 mass combinations for our valence PGBs: 10 degenerate “pions” ($m_x = m_y$) and 45 non-degenerate “kaons” ($m_x \neq m_y$).

To give a more precise indication of the size of our valence-quark masses, we quote in Table VI the values of the valence “physical” down and strange quark masses. These are obtained on each ensemble as follows. The strange mass, m_y , is tuned so that $Y_P = (0.6858 \text{ GeV})^2$, which is the continuum $s - \bar{s}$ mass keeping only quark-connected contractions, as determined by Ref. [38]. The down mass, m_x , is then tuned so that $G = m_{K_0, \text{phys}}^2$. The tunings require extrapolation, and we use the linear extrapolation in quark masses discussed below. For the coarse ensemble, we quote only results for the C3 ensemble, since those on the other coarse ensembles are very similar. Comparing to Table V, we see that our heaviest quark mass is somewhat lighter than the physical strange

quark mass on all ensembles.

TABLE VI. Physical down and strange valence-quark masses (in lattice units).

Ensemble	am_d	am_s
C3	0.00212(2)	0.05174(5)
F1	0.00142(2)	0.03523(5)
S1	0.00102(1)	0.02362(3)

To calculate the four-quark matrix elements needed for B_K we follow the methodology developed in Refs. [39–41]. We place wall-sources composed of U(1)-noise at timeslices t_1 and $t_2 > t_1$, each of which are used to produce both a valence quark and antiquark. Boundary conditions are periodic in all directions. The sources consist of random phases for each color index on each site on the timeslice, and have the property of selecting only the kaon having the desired taste P (i.e. with quantum numbers $\gamma_5 \otimes \xi_5$) and having zero spatial momentum. One source produces a valence kaon, the other an antikaon, and these are contracted together in a gauge-invariant four-quark operator at an intermediate time t . These operators involve quark and antiquark fields spread out over 2^4 hypercubes, and thus live on the two timeslices t and $t+1$. They are made gauge invariant using products of HYP-smear links along the shortest paths joining quarks and antiquarks, averaged over all such paths. The operators are then summed over spatial directions so as to improve the signal.

For completeness, we give expressions for the operators that we use. The four-fermion operator is, at tree-level,

$$\mathcal{O}_K = \sum_{j=1,4} \mathcal{O}_j^{\text{latt}} \quad (47)$$

$$\mathcal{O}_1^{\text{latt}}(t) = \sum_{\vec{y}} \sum_{\mu} [V_{\mu} \times P][V_{\mu} \times P]_I(\vec{y}, t) \quad (48)$$

$$\mathcal{O}_2^{\text{latt}}(t) = \sum_{\vec{y}} \sum_{\mu} [V_{\mu} \times P][V_{\mu} \times P]_{II}(\vec{y}, t) \quad (49)$$

$$\mathcal{O}_3^{\text{latt}}(t) = \sum_{\vec{y}} \sum_{\mu} [A_{\mu} \times P][A_{\mu} \times P]_I(\vec{y}, t) \quad (50)$$

$$\mathcal{O}_4^{\text{latt}}(t) = \sum_{\vec{y}} \sum_{\mu} [A_{\mu} \times P][A_{\mu} \times P]_{II}(\vec{y}, t), \quad (51)$$

where \vec{y} is a vector labeling spatial hypercubes (so that its components are all even). The operators are composed of hypercube-based bilinears, each denoted by $[S \times F]$, where S is the spin and F the taste. The subscripts I and II distinguish the two ways in which the color indices in the bilinears are connected by products of links. A complete definition of these operators is given in Ref. [22]. At one-loop order, the coefficient multiplying each operator differs from unity, as shown in Eq. (59) below.

The four-quark matrix element should be independent of t when $t_1 < t < t_2$, as long as the operator is far

enough from each source to remove contamination from excited states. This is because we always use a kaon and antikaon with the same mass. To obtain B_K we then divide this matrix element by that obtained via the “vacuum saturation” approximation, as shown in Eq. 5. Explicitly, the ratio of correlation functions we use is

$$B_K(t) = \frac{2\langle W(t_1)\mathcal{O}_K(t)W(t_2)\rangle}{(8/3)\langle W(t_1)[A_0 \times P](t)\rangle\langle [A_0 \times P](t)W(t_2)\rangle}, \quad (53)$$

where W represents a wall source, and the factor of 2 in the numerator accounts contractions present in the continuum which are absent for our lattice operator [40]. This ratio should be also be independent of t away from the sources, and cancels the coupling of the wall sources to the (anti)kaons. Thus our aim is to choose the sources so that there is a clear plateau, and then fit B_K to a constant in this region.

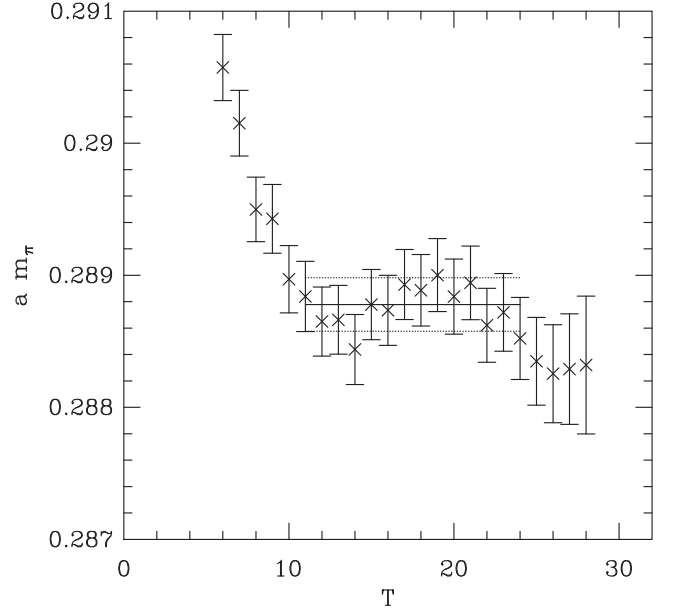


FIG. 2. Pion effective mass versus time from wall-source. Parameters are $am_x = am_y = 0.025$ on the C3 coarse ensemble. The fit is described in the text.

To gauge how far apart to place the sources, we look at the two-point correlator from the wall-source (say at $t = 0$) to the taste- P axial current (i.e. with quantum numbers $\gamma_4\gamma_5 \otimes \xi_5$) summed over the spatial time-slices at times t and $t+1$. This is one of the two factors in the denominator of B_K (and also shows how we obtain the taste- P kaon mass). We show an example of the resulting effective mass in Fig. 2. This is defined by equating the correlator at times t and $t+1$ to the form

$$f(t) = Z[\exp(-m_{\pi}(t+0.5)) - \exp(-m_{\pi}(T-t-0.5))], \quad (54)$$

which includes both a direct and a “wrap-around” exponential (with T the lattice extent in the time direction). The plot is for a pion on one of the coarse ensembles with

a mass close to that of the physical kaon. We observe a noticeable contamination from the excited states up to $t \approx 10$, and that the signal degrades for $26 \lesssim t$. Also shown is a fit of the correlator to the form (54) in the plateau region.

From this and similar plots, we know that on the coarse lattices we need to place our four-fermion operator at least 10 timeslices from both sources, i.e. that $\Delta t = t_2 - t_1 > 20$. Here we assume that the coupling of the four-quark operator to excited states is comparable to that of the axial current—a reasonable assumption given that the vacuum saturation approximation turns out to be good to within a factor of two. The source separation cannot be too large, however, because the signal will then be contaminated by wrap-around contributions arising from our use of a periodic lattice. Thus we have chosen Δt to be somewhat larger than the minimum, but not too much larger. Our specific choices are given in Table VII. We have held the physical distance between the sources, $\Delta t/a$, roughly constant as the lattice spacing is reduced. The table also gives the parameters that determine the fitting range: we fit from $t = t_1 + t_L$ to $t = t_1 + t_R = t_2 - t_L - 1$. Note that our fit ranges are symmetrical with respect to the two wall sources since the four-quark operator resides on timeslices t and $t + 1$.

TABLE VII. Choices for the wall-source separation, Δt , and its ratio to the temporal length of the lattices, T , as well as the parameters determining the fitting range.

a (fm)	Δt	$\Delta t/T$	t_L	t_R
0.12	26	0.41	10	15
0.09	40	0.42	13	26
0.06	60	0.42	22	37

We can estimate the systematic error due to wrap-around contributions as follows. These contributions have the kaon from one source propagating the “wrong” way around the lattice, so that the matrix element which occurs is $\langle K_0 K_0 | \mathcal{O} | 0 \rangle$ or its conjugate. At leading order in continuum ChPT, this matrix element has the same magnitude (and opposite sign) as the desired matrix element $\langle K_0 | \mathcal{O} | \bar{K}_0 \rangle$. Since the wrap-around effect is small, when estimating its size we ignore the differences between these two matrix elements due to higher order SChPT effects. Then the contamination is simply given by the ratio of the kaon propagators in the two cases. Adding the two contributions involving a single (anti)kaon wrapping around the lattice, we find that the ratio of contamination to signal is

$$R_K(t) = 2 \exp\{-am_K(T - \Delta t)\} \times \cosh\{2am_K[(t - t_1) - (\Delta t - 1)/2]\}. \quad (55)$$

This is suppressed by the overall exponential factor, but grows on either side of its minimum (which occurs midway between the sources) as a cosh with an exponent containing $2am_K$. To give an idea of the size of the effect we give two examples of the value of R_K at the edge

of the fitting range (where it is maximal). On the coarse lattices, with $T = 64$, $\Delta t = 26$ and $t - t_1 = t_L - t_1 = 10$ we find $R_K(t) = 0.015$ for the lightest pion ($am_{xy:P} = 0.1342$ for $am_x = am_y = 0.005$), while $R_K(t) = 1.2 \times 10^{-4}$ for the lightest kaon used in the SU(2) chiral fit ($am_{xy:P} = 0.2749$ for $am_x = 0.005$ and $am_y = 0.04$). The latter result is most relevant for an estimate of the size of the induced error in our final result for the physical kaon (which has a mass $am_K^{\text{phys}} \approx 0.3$ on the coarse lattices). This is because, whether we use the SU(3) or SU(2) fits, the most important points are those closest to the place to which we are extrapolating. Thus we conclude that the wrap-around systematic is negligible for B_K , although it would be significant (at the few percent level) were we to quote a result for B_0 (B_K in the chiral and continuum limits).

Based on the autocorrelation lengths for light hadron properties, the lattices in the MILC ensembles are somewhat correlated [19]. Because of this, we attempt to reduce correlations between our measurements of B_K by randomly choosing the overall source position, t_1 , on each configuration (with t_2 fixed to $t_1 + \Delta t$). For three ensembles (C3, C3-2 and C4) we have increased our statistics by doing multiple measurements on each lattice (as shown in Table III). We chose the t_1 values for these measurements randomly and also set the random number seeds for the U(1) noise sources randomly.

Using these enlarged data-sets, we have done a preliminary study of autocorrelations by seeing how the error changes when we bin configurations together. This has been done for a pion two-point correlator, for which the errors are smaller than those for B_K . The results for ensemble C3 are presented in Ref. [16]. We find that the errors increase somewhat as the bin size increases, reaching a plateau for bins of size 3 or above, in which the errors are about 20% larger than those obtained treating each configuration as independent. This increase is only visible after we use the larger sample (9 measurements per configuration).

We take this result as indicating that autocorrelations are a small effect. In particular, because our statistical errors turn out to be smaller than the systematic errors, a 20% increase in the former would have little impact on our final result. Thus we have ignored the possible impact of autocorrelations in this paper.

Typical results for one-loop renormalized B_K , along with the fitting range and resulting value and error, are shown in Fig. 3. This figure shows the dependence on the lattice spacing when we use a kaon composed of degenerate quarks but with a mass close to that of the physical kaon. The data have a similar profile for all three lattice spacings, showing contamination from excited states near the boundary but a reasonable plateau that extends further out than our fit region. The smaller size of the errors on the coarse lattice reflects the larger sample (9 measurements on each lattice).

The effect of varying the quark masses (at fixed lattice spacing) is illustrated by Fig. 4. The (blue) diamonds

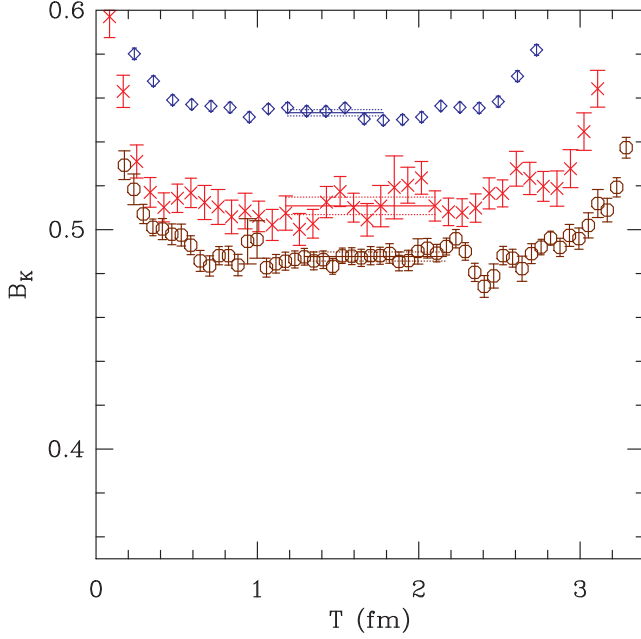


FIG. 3. $B_K(\mu = 1/a)$ as a function of $T = t - t_1$. (Blue) diamonds are from the coarse ensemble C3, with $am_x = am_y = 0.025$; (red) crosses are from the fine ensemble F1, with $am_x = am_y = 0.015$; and (brown) octagons are from the superfine ensemble S1, with $am_x = am_y = 0.009$. The result of a fit to a constant over the range given in Table III is shown by the horizontal lines. Note that the T is given in fermis.

show the same data as in Fig. 3, with degenerate quarks and an approximately physical kaon mass. The (red) crosses show the change when $m_y - m_x$ is maximized (in our data set) with $m_x + m_y$ fixed (so that the kaon mass is almost the same). One sees that the central value increases a little, and that the errors are significantly larger. The (brown) octagons shows what happens if both m_x and m_y are reduced to our smallest value, so that the resulting “kaon” is closer to a pion (having a mass ≈ 220 MeV). Here the central value is significantly reduced, and the errors are much larger. Fortunately, our results from light “kaons” play little role in determining the extrapolation to the physical kaon.

In these figures, and in all subsequent analysis, errors are obtained using the jackknife procedure. This is true both for the errors on individual points and for the errors in fit parameters. When doing fits to multiple timeslices, and when fitting to multiple quark masses (as required for the chiral fits discussed below) we do not use the (inverse of the) full covariance matrix, which we have found to lead to instabilities, but rather keep only the diagonal elements. This means that we cannot quantitatively judge the goodness of fit, since the χ^2 is generically underestimated. Assuming, however, that the fits are reasonable, the jackknife errors should be reliable.

We close this subsection by describing how we determine the values of $m_{xy:B}$ needed as inputs to the SchPT

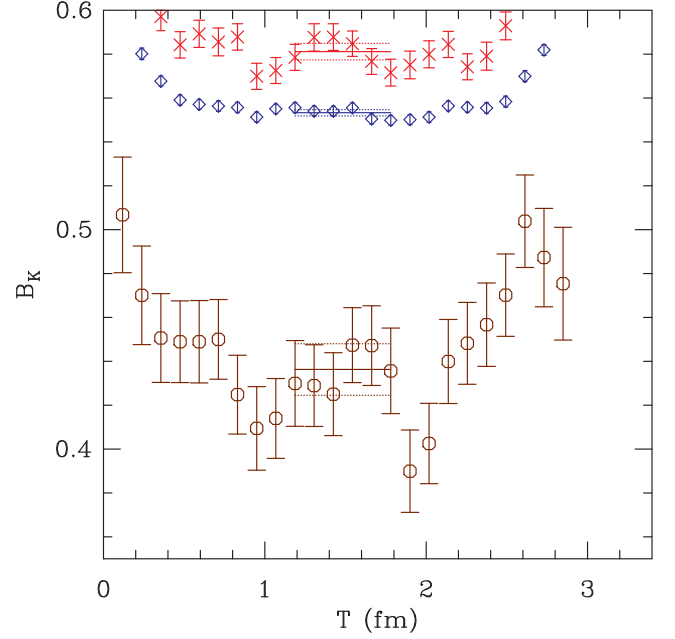


FIG. 4. As for Fig. 3, except that all data is from the C3 ensemble. (Blue) diamonds show results for $am_x = 0.025$ and $am_y = 0.025$, (red) crosses for $am_x = 0.005$ and $am_y = 0.045$, and (brown) octagons for $am_x = 0.005$ and $am_y = 0.005$.

fits to B_K . For the taste-P meson, we measure the mass directly using our wall sources. We have four sets of two-point correlation functions:

$$C_1(t) = \langle A_4(t)P(t_1) \rangle, \quad (56a)$$

$$C_2(t) = \langle A_4(t)P(t_2) \rangle, \quad (56b)$$

$$C_3(t) = \langle P(t)P(t_1) \rangle, \quad (56c)$$

$$C_4(t) = \langle P(t)P(t_2) \rangle. \quad (56d)$$

We determine the Goldstone pion masses by fitting these four sets of data individually and averaging the results.

We have calculated the masses of PGBs having all other tastes on most of the ensembles. The methodology is explained in Refs. [30, 42]. Specifically, we use 8 “cubic wall sources” per configuration, and “Golterman-style” sink operators. Results for the best-measured tastes are shown in Figs. 5 and 6 for a coarse and the fine ensemble, respectively. The nondegenerate points are also consistent within errors with the observed linear behavior, though we do not show them for the sake of clarity. The results for the other tastes have masses that are completely consistent with the $SO(4)$ symmetry expected at LO. Given this, we choose to assume exact $SO(4)$ symmetry for the masses used in the ChPT expressions, taking the values from the best measured tastes (such as those in the Figures).

The LO prediction of Eq. (6) has the slopes being the same for all tastes. NLO effects will, however, lead to a difference between the slopes. We find that the slopes do have statistically significant differences, but that these differences are very small, a few percent or less [42]. In

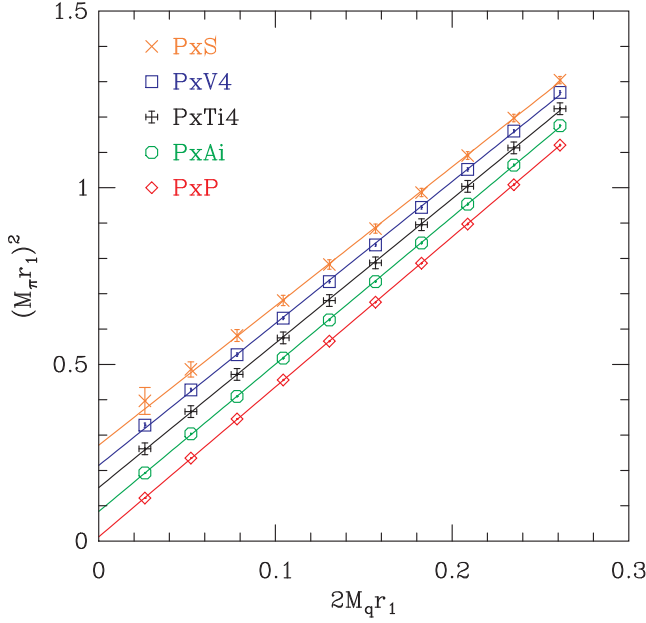


FIG. 5. $m_{xy}^2 r_1^2$ versus $(m_x + m_y)r_1$, for states with $m_x = m_y$, and tastes (from bottom to top) ξ_5 , $\xi_i \xi_5$, $\xi_i \xi_4$, ξ_4 and 1. r_1 is the modified Sommer-parameter, whose value is $2.619a$. The fits are discussed in the text. Results are for the coarse ensemble C3.

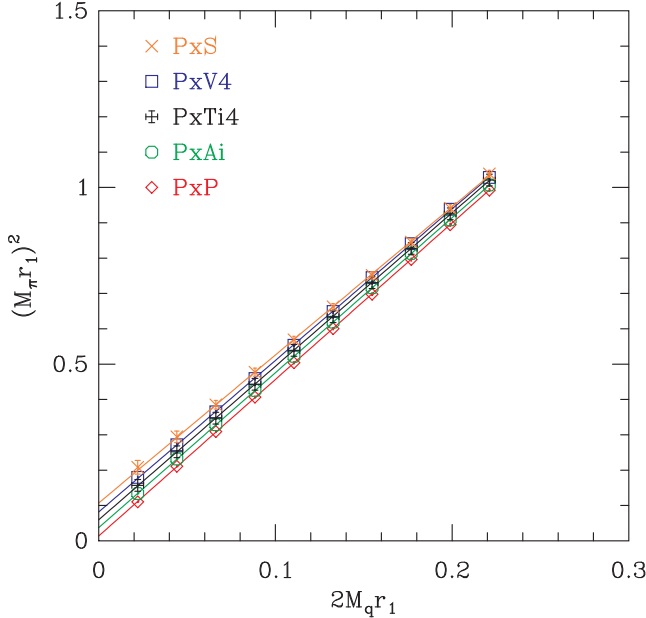


FIG. 6. As in Fig. 5, but for the fine ensemble F1. Here $r_1 = 3.701a$.

practice we do linear fits allowing the slopes to differ (i.e. allowing b_1 to depend on taste B), and from these determine the intercepts Δ_B . We then reconstruct the

PGB masses that we use in chiral loops using

$$K_B = G + \Delta_B - \Delta_P, \quad (57a)$$

$$X_B = X_P + \Delta_B - \Delta_P, \quad (57b)$$

$$Y_B = Y_P + \Delta_B - \Delta_P. \quad (57c)$$

This choice effectively assumes that the slopes are equal to that of the taste P mesons. This approach reproduces the actual masses to within a few percent or better, and is, in any case, theoretically consistent as the LO masses can be used in loops. Put differently, the inclusion of the small differences between slopes in the chiral logarithms would lead to an analytic contribution of NNLO, i.e. of higher order than our expressions.

We proceed in the manner just described on all ensembles except the large-volume coarse ensemble (C3-2), the fine ensemble (F2), and the superfine ensemble (S1). On these three ensembles we have not measured the full taste spectrum, due to the high computational cost. We estimate the taste splittings on these ensembles as follows. For (C3-2) we simply use the taste splittings from the smaller lattices with the same quark masses [ensemble (C3)]. For (F2) we assume the same taste splittings as for the other fine ensemble, (F1), based on the fact that taste splittings on the coarse lattices are almost independent of the light sea-quark mass. Finally, for the superfine ensemble, we obtain the Δ_B by extrapolating from the coarse and fine lattices. These quantities are expected to vanish in the continuum limit as $a^2 \alpha^2$. We thus fit our results from the C3 and F1 ensembles to

$$\Delta_B(a) = b_1 a^2 \alpha_s^2(\mu = 1/a) \quad (58)$$

with α_s determined using 4-loop evolution from the Particle Data Group value at $\mu = M_Z$. The result is satisfactory, as shown in Fig. 7, and leads to the values collected in Table VIII. Since the taste splittings on the superfine lattice are so small, the uncertainty introduced by the need to do this extrapolation has a negligible impact on our chiral fitting.

TABLE VIII. Results for valence taste-splittings, Δ_B (GeV²), on ensembles C3, F1 and S1. Those on C3 and F1 are obtained by calculating PGB masses using cubic wall sources (see Ref. [42]). Those on S1 are obtained by extrapolating using Eq. (58).

a (fm)	0.12 (C3)	0.09 (F1)	0.06 (S1)
Δ_A	0.02905(59)	0.00912(29)	0.003487(60)
Δ_T	0.0565(13)	0.01753(49)	0.00666(12)
Δ_V	0.0819(18)	0.02612(68)	0.00977(16)
Δ_S	0.1052(88)	0.0314(33)	0.01219(80)

B. Matching factors

To convert the results of our lattice calculation into physical observables, we need to match lattice and con-

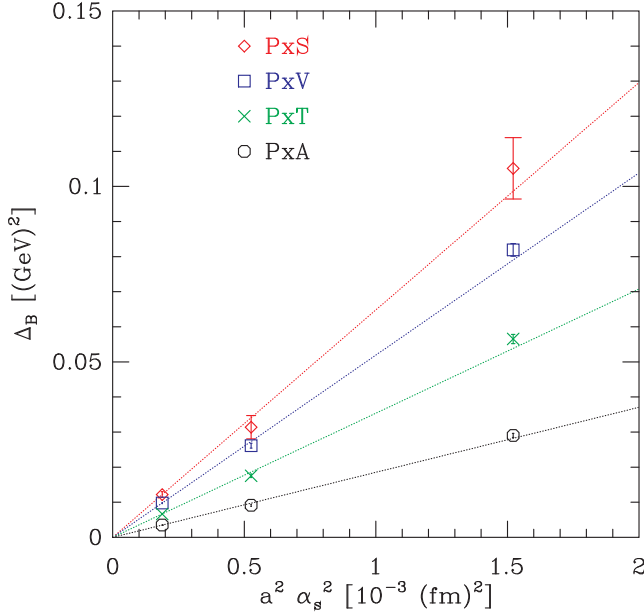


FIG. 7. Δ_B from ensembles C3 and F1 (the rightmost two sets of points). These are extrapolated, as described in the text, to obtain the results for the S1 ensemble (leftmost set of points). Numerical values are given in Table VIII.

tinuum operators. The latter we define in the conventional way using naive dimensional regularization with $\overline{\text{MS}}$ subtraction. For the matching, we use one-loop perturbation theory, which leads to the general form

$$O_i^{\text{cont}}(\mu) = \sum_j \left[\delta_{ij} - \frac{\alpha(q^*)}{4\pi} (\gamma_{ij} \log(a\mu) + c_{ij}) \right] O_j^{\text{latt}} \quad (59)$$

Here superscripts “cont” and “latt” indicate continuum and lattice operators, respectively, γ_{ij} is the anomalous dimension matrix, the c_{ij} are finite constants, and μ is the renormalization scale for the continuum operator. We choose the coupling constant here and below to be that defined in the continuum $\overline{\text{MS}}$ scheme (which we evaluate numerically using four-loop running from m_Z). The scale at which it is evaluated, q^* , is undetermined by one-loop matching.

In our calculation we are interested in a single continuum operator, so i takes one value, but the matching involves many lattice operators, labeled by j . The operators which appear are listed in Ref. [22]. The values of c_{ij} for our HYP-smear operators, using the appropriate Symanzik-improved gauge action, have been calculated by three of us and will be presented in Ref. [43].

When using this matching formula we do not, in fact, include all the lattice operators which appear at one-loop, but instead keep only those with external tastes P . There are four of these, and they are listed above in Eqs. (48–51). Dropping the other operators leads to the $O(\alpha/\pi)$ truncation errors discussed earlier.

Having matched to the continuum operator at an intermediate scale μ , we then evolve using the continuum

renormalization group (RG) down to a canonical scale $p = 2 \text{ GeV}$. We do so using the two-loop formula

$$B_K(\text{NDR}, p) = \frac{[1 - \frac{\alpha(\mu)}{4\pi} Z]}{[1 - \frac{\alpha(p)}{4\pi} Z]} \left(\frac{\alpha(p)}{\alpha(\mu)} \right)^{d^{(0)}} B_K(\text{NDR}, \mu) \quad (60)$$

$$Z = \frac{\gamma^{(1)}}{2\beta_0} - d^{(0)} \frac{\beta_1}{\beta_0}$$

$$d^{(0)} = \frac{\gamma^{(0)}}{2\beta_0}.$$

Expressions for the β_i and $\gamma^{(i)}$ are given, for example, in Ref. [44]. In this way we obtain $B_K(\text{NDR}, p = 2 \text{ GeV})$.

In order to implement this matching and running we have to choose values for μ and q^* . The result depends on these choices because we have truncated the matching at one-loop and the running at two-loops. In both cases the error we make is of $O(\alpha^2)$. Based on the arguments of Ref. [45], we use “horizontal matching” in which $q^* = \mu$. The choice of q^* should be made so as to minimize higher-order terms in perturbation theory, which requires that q^* be a typical scale of momenta flowing in loops. We simply take $q^* = 1/a$. It is possible to make a better-informed estimate using the one-loop integrands themselves, but we have not yet attempted this.

Finally, we recall that one can also define the RG invariant quantity \hat{B}_K ,

$$\hat{B}_K = [1 - \frac{\alpha(\mu)}{4\pi} Z] \left(\frac{1}{\alpha(\mu)} \right)^{d^{(0)}} B_K^{\text{NDR}}(\mu), \quad (61)$$

which is often used in the continuum literature. This quantity has the apparent advantage that all truncation errors are manifestly proportional to $\alpha(\mu)^2$, with the scale μ being naturally chosen of $O(1/a)$ (and in our case, chosen to be exactly $1/a$). Thus these errors vanish (albeit logarithmically) in the continuum limit. By comparison, using $B_K(\text{NDR}, 2 \text{ GeV})$ apparently leads to truncation errors which have a nominal size of $\alpha(2 \text{ GeV})^2$, which does not vanish in the continuum limit. This is, however, misleading, in that if different lattice calculations use the same truncated formula to run to 2 GeV , they are in effect comparing \hat{B}_K , so the real truncation errors in the comparison are $\sim \alpha(1/a)^2$.

C. SU(3) fitting: Strategy

We would like to fit our results for B_K to the functional form of Eq. (21). For a given ensemble (fixed a and sea-quark masses) the fit form has 15 parameters.⁷

⁷ There are 15 rather than 16 parameters because H_5 is a constant for fixed sea-quark masses, and so can be absorbed into b_1 (the

Despite the fact that we have 10 degenerate and 45 non-degenerate data points on each ensemble, we find a direct fit to Eq. (21) to be very difficult. Here we explain why and describe our strategies for dealing with this problem.

We begin by sorting the 15 terms into 4 categories, according to the following criteria:

- Does the term survive in the continuum limit, i.e. does it correspond to a term in continuum ChPT? Contributions from discretization errors (taste-breaking or otherwise) and from truncation errors do not have such a correspondence.
- Does the term contribute to B_K in the degenerate case of $m_x = m_y$, or does it vanish in that limit?

The result of this categorization is shown in Table IX.

TABLE IX. Classification of contributions, H_i , to the NLO SU(3) SchPT expression for B_K , Eq. (21), when fitting at a fixed a and sea-quark masses. Terms in the “cont” column have a corresponding continuum form, while those labeled “non-cont” do not. Terms in the row labeled “deg” contribute both for $m_x = m_y$ and $m_x \neq m_y$, while those in the “non-deg” row contribute only for $m_x \neq m_y$. Numbers in the table give the subscript, i , of the fit form H_i . Numbers underlined indicate those H_i which are kept in our final SU(3) fits.

	cont	non-cont
deg	<u>1</u> , <u>2</u> , <u>3</u>	6, 7, 8, 9, <u>10</u>
non-deg	<u>4</u>	11, <u>12</u> , 13, <u>14</u> , 15, 16

From Table IX, we observe that the fitting functional is somewhat simplified if we fit only to degenerate data points. There are then three “continuum-like” terms, H_{1-3} , and five terms which are lattice artifacts, H_{6-10} . Since we have 10 degenerate data points, we can, in principle, fit them to a functional form composed of 8 terms. In practice, this turns out to be very difficult, primarily because the functional forms of the five lattice-artifact terms are very similar for our range of PGB masses. This is not unexpected: the five functions have the same form [that of $F^{(4)}$ defined in Eq. (A9)], but differ in the taste of the PGB whose mass appears. Thus one expects the functions to be similar except at the lightest quark masses where the taste-splittings are comparable to the masses themselves (see Fig. 5). This is borne out by direct numerical evaluation. The similarity of the functions becomes more pronounced as one approaches the continuum limit, because of the reduction in taste splittings. Indeed, on the superfine lattices, one can argue that the $\mathcal{O}(a^2)$ effects should be treated as of NLO, and thus that

the difference between these five functions is really an effect of NNLO.

In light of these considerations, we approximate all five functions by $F_T^{(4)}$, i.e. the function in which the mass is that of the taste-T PGB. We choose the tensor taste as its mass is roughly the average of those of all tastes. We stress that this is not a systematic procedure, since if $\mathcal{O}(m) \sim \mathcal{O}(a^2)$, as is the case for the smallest quark masses on the coarse lattices, it is *not* legitimate to expand $\log(X_B)$ about $X_B = X_T$ and treat the difference as of higher order. Our procedure is simply a phenomenological way of resolving the presence of nearly flat directions in the fit function.

With this approximation, the fitting function for the degenerate data-set collapses to

$$f_{\text{th}}^{\text{deg}} = \sum_{i=1}^4 c_i F_i \quad (62)$$

where $F_1 = H_1$, $F_2 = H_2$, $F_3 = H_3$ [with the H_i as defined in Eqs. (22-25)] and $F_4 = H_{10} = F_T^{(4)}$. Fits to this 4-parameter form are stable, and we use them as the first stage of most of our fitting strategies.

The fit to (62) can also be constrained given our prior knowledge of the size of the coefficients. We expect from continuum ChPT power-counting that $c_1 \sim c_2 \sim c_3 \sim \mathcal{O}(1)$. For $c_4 = b_{10}$ we have two possible estimates, given in Eq. (35), depending on whether discretization or truncation errors dominate. One can systematically include this prior information in the fitting using the Bayesian method [46], as discussed in the next subsection.

We now turn to the remaining 45 non-degenerate data points. As can be seen from Table IX, including these points in the fit requires the introduction of one additional term with a non-vanishing continuum limit, and six terms arising from lattice artifacts. Once again, however, directly fitting to all these terms is difficult because some of the functions are similar. Thus we choose again to approximate the fitting function.

We first observe that $H_{11} = F_V^{(5)}$ [given in Eq. (A10)] is numerically close to twice $H_{12} = F^{(6)}$ [given in Eq. (A11)]. This proportionality is exact if taste-splittings are ignored. Thus we choose to drop H_{11} and keep H_{12} as a surrogate for both these functions.

Next we note that $H_{13} = F_V^{(1)}$ is very similar to $H_{14} = F_A^{(1)}$ —differing only in the taste of the PGBs whose masses appear in the function. Thus we choose to keep only H_{14} from this pair.

Finally, we come to $H_{15} = F_V^{(2)}$ and $H_{16} = F_A^{(2)}$. These are also numerically similar, but, more importantly, their coefficients are proportional to δ_A^{MA2} and δ_V^{MA2} , quantities that we have argued are substantially suppressed (see Sec. II A). Thus we consider these contributions to be of NNLO and simply drop them.

The outcome of these considerations is that the functional form we use when fitting our entire data-set con-

coefficient of H_1). Doing so leads to additional chiral logarithms multiplied by quark masses, but these are of NNLO, and thus irrelevant to our NLO expression. Of course, having absorbed b_5 into b_1 , we must subsequently take into account the resulting sea-quark mass dependence of b_1 . This we do in Sec. III F.

tains 7 terms:

$$f_{\text{th}} = \sum_{i=1}^7 c_i F_i \quad (63)$$

where the first 4 terms are the same as in Eq. (62) and the rest are $F_5 = H_4$ (the analytic term), $F_6 = H_{12} = F^{(6)}$ and $F_7 = H_{14} = F_A^{(1)}$. We expect $c_5 = \mathcal{O}(1)$ and $c_6 \sim c_4$, while $c_7 = b_{14}$ is given by the estimates in Eq. (40). In some of our fits we impose the latter two estimates using the Bayesian method.

D. SU(3) fitting: Implementation

In this section, we use our fits to the C3 and S1 ensembles as examples to explain our detailed strategy and the results.

We first fit the degenerate data to the 4-parameter form of Eq. (62) with no constraints. We label this fit “D-U”, with D for degenerate and U for unconstrained.⁸

We show the result of this fit on the C3 and S1 en-

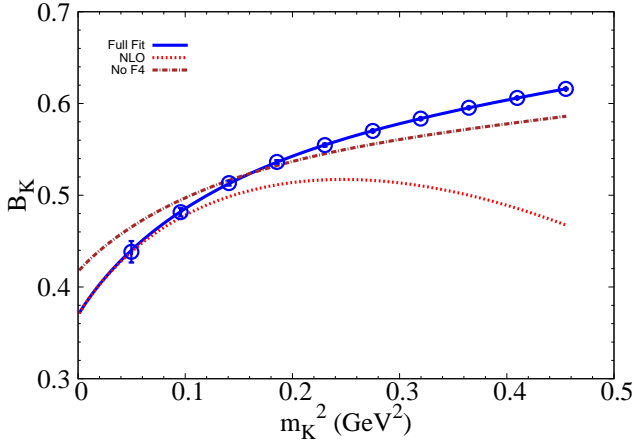


FIG. 8. $B_K(\text{NDR}, \mu = 1/a)$ vs. m_K^2 for degenerate kaons ($m_x = m_y$), on the C3 ensemble. A fit of type D-U is shown by the solid (blue) curve. The dotted (red) curve labeled “NLO” shows the impact of dropping the NNLO term. The dashed (brown) curve labeled “No F4” shows the impact of dropping the F_4 term.

sembles in Figs. 8 and 9, respectively. These ensembles have, to good approximation, the same sea-quark masses and thus differ mainly in the value of a . The fits give a good representation of the data, as is the case also on the other ensembles (not shown). The parameters from the D-U fits are collected in Table X.⁹ We stress again that the χ^2/dof uses only diagonal elements of the covariance

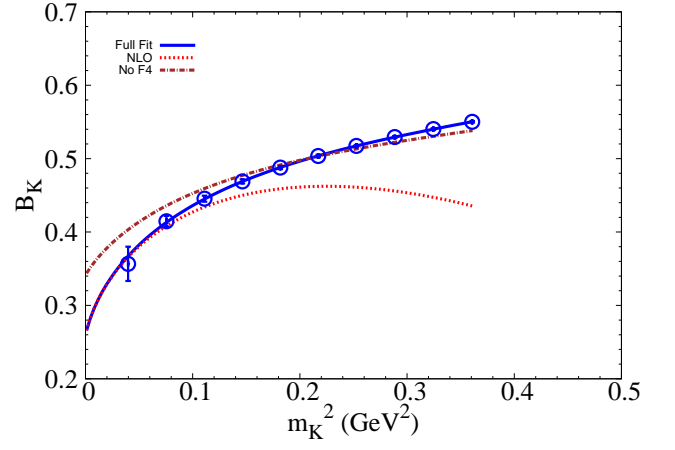


FIG. 9. $B_K(\text{NDR}, \mu = 1/a)$ vs. m_K^2 for degenerate kaons on the S1 ensemble showing a D-U fit. Notation as in Fig. 8

matrix and thus is expected to be much less than unity. In fact, a value approaching unity indicates a poor fit.

TABLE X. Parameters of D-U fits. The c_i are defined in Eq. (62). χ^2/dof represents χ^2 per degree of freedom calculated using only diagonal elements of the covariance matrix.

ID	c_1	c_2	c_3	$c_4(\text{GeV}^2)$	χ^2/dof
C1	0.283(41)	0.88(40)	0.24(41)	0.0018(19)	0.004(20)
C2	0.351(42)	0.18(41)	0.85(41)	-0.0022(19)	0.012(17)
C3	0.339(13)	0.37(13)	0.72(13)	-0.0008(6)	0.041(54)
C4	0.327(13)	0.47(13)	0.60(13)	-0.0005(6)	0.051(59)
C5	0.286(32)	0.84(30)	0.22(30)	0.0013(14)	0.08(11)
F1	0.349(39)	0.05(33)	1.06(40)	-0.0012(11)	0.10(12)
F2	0.356(32)	0.01(27)	1.13(33)	-0.0013(9)	0.11(13)
S1	0.323(25)	0.18(21)	0.88(29)	-0.0005(6)	0.05(12)

We find that c_{1-3} are of $\mathcal{O}(1)$, indicating that the chiral expansion is behaving as expected. The convergence of the ChPT series is illustrated in the figures. At LO, B_K is a constant. The NLO form is shown by the (brown) dashed curves, and has curvature due to the chiral logarithms in F_1 and F_4 . Our phenomenological NNLO term makes up the difference between the full fit and the NLO form. The convergence is reasonable for $m_K \lesssim m_K^{\text{phys}}$, (e.g. on C3 we find LO:NLO:NNLO $\approx 1 : 0.5 : 0.12$ for $m_K \approx m_K^{\text{phys}}$), but clearly poor for our heaviest kaon. We stress, however, that, because we do not have the complete NNLO expression, this analysis of convergence is only approximate.

Since c_{1-3} are continuum parameters, and are dimensionless, we would expect approximate consistency between their values in all the fits, up to a modest dependence on lattice spacing and on sea-quark masses. This expectation holds within errors—although we note that the errors in c_2 and c_3 are quite large.

The coefficient c_4 multiplies the sole NLO discretization/truncation term in our fit function. We find that

⁸ This fit is called “D-T3” in our previous publications.

⁹ Ensemble C3-2 is only used for an estimate of finite-volume errors, as described in Sec. III H.

our fits do not require such a term—it is consistent with zero on all ensembles. The range of values of c_4 allowed by the fits (as indicated by the errors) are such, however, that this term can make a noticeable contribution. This is illustrated in the figures, where the contribution of the c_4 term is given by the difference between the full fits and the “No F4” curves. It is seen to be relatively small ($< 10\%$) over the entire mass range on the C3 ensemble, and in particular is considerably smaller than continuum-like NLO terms (and comparable to the NNLO terms) at m_K^{phys} . As expected, its contribution on the superfine ensemble is generally much smaller than on the coarse ensemble. These results imply that we have been conservative in treating this term as of NLO in our power-counting.

Another way of seeing this is to compare the values of c_4 (and their errors) to the magnitudes expected from power counting. As noted above, these range from

$$|c_4| = \mathcal{O}(a^2) \approx \Lambda_{\text{QCD}}^2 (a\Lambda_{\text{QCD}}^2)^2$$

$$\approx \begin{cases} 0.0031 \text{ GeV}^2 & \text{on the coarse lattices,} \\ 0.0015 \text{ GeV}^2 & \text{on the fine lattice,} \\ 0.00075 \text{ GeV}^2 & \text{on the superfine lattice.} \end{cases} \quad (64)$$

(where we have used $\Lambda_{\text{QCD}} = 0.3 \text{ GeV}$) to

$$|c_4| = \mathcal{O}(\alpha^2) \approx \Lambda_{\text{QCD}}^2 \alpha^2$$

$$\approx \begin{cases} 0.010 \text{ GeV}^2 & \text{on the coarse lattice} \\ 0.0069 \text{ GeV}^2 & \text{on the fine lattice} \\ 0.0050 \text{ GeV}^2 & \text{on the superfine lattice} \end{cases} \quad (65)$$

(where we have evaluated α at the scale $\mu = 1/a$). Our results for $|c_4 \pm \delta c_4|$ are at or below the *smaller* of these estimates on all ensembles.

We have also considered the effect of constraining c_4 using Bayesian fits. Specifically, following Ref. [46], we augment the usual χ^2 with an additional term:

$$\chi_{\text{aug}}^2 = \chi^2 + \chi_{\text{prior}}^2 \quad (66)$$

$$\chi_{\text{prior}}^2 = \frac{(c_4 - a_4)^2}{\tilde{\sigma}_4^2}. \quad (67)$$

Since we have no prior information on the sign of c_4 we set $a_4 = 0$. We do have prior information on the expected size of $c_4 = b_{10}$, given in Eq. (35), and we choose $\tilde{\sigma}_4$ accordingly. In particular we consider two Bayesian fits, one assuming that discretization errors dominate [$\tilde{\sigma}_4 \approx \Lambda_{\text{QCD}}^2 (a\Lambda_{\text{QCD}}^2)^2$], the other assuming that truncation errors dominate [$\tilde{\sigma}_4 \approx \Lambda_{\text{QCD}}^2 \alpha^2$]. We label these “D-B1” and “D-B2”, respectively.¹⁰ The numerical values of $\tilde{\sigma}_4$ are given in Eqs. (64) and (65) above.

We expect that adding these constraints will have little impact on the fits, given that the unconstrained fits yield

values of c_4 and δc_4 that are mostly consistent even with the stronger of the constraints. This is borne out by the results of the fits. We show the results for the D-B1 case in Table XI. There is no significant change in c_{1-3} , but the central values, and the errors, in c_4 have been reduced on some ensembles (with a corresponding increase in the augmented χ^2). As expected, the less constrained D-B2 fits are almost identical to the D-U fits, so we do not show them.

TABLE XI. Parameters of D-B1 fits.

ID	c_1	c_2	c_3	$c_4 (\text{GeV}^2)$	$\chi_{\text{aug}}^2/\text{dof}$
C1	0.300(25)	0.71(23)	0.42(23)	0.0011(12)	0.044(72)
C2	0.328(23)	0.39(22)	0.61(21)	-0.0013(12)	0.068(99)
C3	0.338(12)	0.38(12)	0.70(12)	-0.0008(6)	0.053(70)
C4	0.326(12)	0.48(12)	0.59(12)	-0.0005(6)	0.057(54)
C5	0.295(23)	0.76(21)	0.31(21)	0.0009(10)	0.11(15)
F1	0.327(21)	0.24(17)	0.81(20)	-0.0007(6)	0.17(22)
F2	0.338(21)	0.15(17)	0.93(20)	-0.0009(6)	0.20(22)
S1	0.311(13)	0.28(11)	0.73(15)	-0.0003(3)	0.11(21)

We now turn to fits to our full data set, i.e. to all 55 degenerate and non-degenerate points. We first fit to the functional form (63) without constraints, labeling the fits “N-U” (with N for non-degenerate, U for unconstrained).¹¹ The parameters of the N-U fit for all ensembles are given in Table XII, and the quality of the fit is illustrated by Figs. 10 and 11. To better display the fit, we plot B_K versus X_P rather than $G = m_K^2$. This has several advantages: it spreads the points out, giving a sense of the quality of the entire fit; it displays the dependence on the light valence quark mass $m_x \propto X_P$, which is the critical parameter which must be extrapolated in order to reach the physical point; and it allows a more direct comparison with the SU(2) “X-fits” presented below. The fits are reasonable—as is the case on all other ensembles. The C3 and C4 ensembles have larger χ^2/dof because the statistical errors are substantially smaller, which brings to light the inadequacies in our partial-NNLO fit form.¹²

We also include in the figure the value of $B_K(NDR, \mu = 1/a)$ extrapolated to the physical quark masses using the fit function, *with all taste-breaking discretization and truncation errors removed*.¹³ This “physical” B_K is plotted at the physical value of X_P . We also

¹¹ This corresponds to N-T2 in our previous publications.

¹² As noted by the MILC collaboration, more precise lattice data requires the use of ChPT expressions of higher order, with the order chosen so the error from the truncation of the series is reduced to the level of the statistical errors.

¹³ In detail, we remove the terms induced by lattice artifacts by setting $c_4 = c_6 = c_7 = 0$, remove all taste splittings in sea and valence mesons ($\Delta_B = 0$), and choose physical valence-quark masses by setting $Y_B = m_{s\bar{s},\text{phys}}^2 = (0.6858 \text{ GeV})^2$ [38], $G = K_B = m_{K_0,\text{phys}}^2$, and $X_B = 2m_{K_0,\text{phys}}^2 - m_{s\bar{s},\text{phys}}^2$ (the lat-

¹⁰ These correspond to D-BT7 and D-BT7-2 in our previous publications.

TABLE XII. Parameters from N-U fits. The c_i are defined in Eq. (63), and χ^2/dof is the uncorrelated χ^2 per degree of freedom. The last column gives the value for $B_K(2\text{GeV})$ that results when we set quark masses to their physical values and remove discretization errors (as described in the text) and then run the resulting $B_K(1/a)$ down to 2 GeV.

ID	c_1	c_2	c_3	$c_4(\text{GeV}^2)$	c_5	$c_6(\text{GeV}^2)$	$c_7(\text{GeV}^4)$	χ^2/dof	$B_K(\mu = 2\text{GeV})$
C1	0.275(38)	0.96(37)	0.15(37)	0.0021(17)	0.050(34)	-0.0054(37)	0.0012(5)	0.0063(80)	0.616(30)
C2	0.351(38)	0.17(37)	0.86(37)	-0.0022(18)	0.103(36)	-0.0051(42)	0.0015(6)	0.0039(38)	0.602(33)
C3	0.331(12)	0.46(12)	0.63(12)	-0.0004(6)	0.057(11)	-0.0025(12)	0.0008(2)	0.135(41)	0.596(10)
C4	0.319(12)	0.56(12)	0.51(12)	-0.0001(6)	0.062(11)	-0.0034(12)	0.0008(2)	0.196(47)	0.603(10)
C5	0.285(29)	0.85(28)	0.21(28)	0.0014(13)	0.069(24)	-0.0052(27)	0.0008(3)	0.040(30)	0.621(22)
F1	0.330(37)	0.21(31)	0.87(38)	-0.0007(10)	0.053(25)	-0.0017(23)	0.0006(3)	0.038(33)	0.564(20)
F2	0.343(30)	0.10(25)	1.00(31)	-0.0010(9)	0.039(18)	0.0000(18)	0.0003(2)	0.034(30)	0.550(16)
S1	0.316(23)	0.23(20)	0.80(26)	-0.0004(5)	0.072(15)	-0.0031(12)	0.0005(2)	0.018(22)	0.581(12)

include physical B_K values in Table XII, and subsequent tables for other fits, except that in the tables we run them to a common scale, $\mu = 2\text{ GeV}$, so as to allow comparison between ensembles.

We stress that we are only able to remove taste-breaking discretization errors because we have used a *staggered* ChPT fit in which such breaking is explicit. We also note that taste-conserving discretization and truncation errors are not removed by this procedure, and must be dealt with separately.

The values of the parameters $c_1 - c_4$ given in Table XII are consistent with those from the fits to the degenerate data alone. This is a non-trivial result, suggesting that our truncation of the full NLO SU(3) expression down to 7 parameters can adequately describe the data. The “continuum non-degenerate” parameter c_5 appears to be non-zero (this is most significant on the high-statistics ensembles C3 and C4), and is consistent across all ensembles. The “lattice non-degenerate” parameters c_6 and c_7 appear also to be non-zero on the high-statistics ensembles C3 and C4, although they are consistent with zero on other ensembles. Our errors are too large to tell whether c_6 and c_7 show the expected decrease as the continuum limit is approached.

The most striking result, however, is that c_5 is very small. It is expected to be of order unity, but turns out to be more than an order of magnitude smaller. We do not know of an explanation for this smallness. Indeed, its value is not of direct physical significance, since it depends on the scale chosen in the chiral logarithms (which we have taken to be $\mu_{\text{DR}} = 0.77\text{ GeV}$). Nevertheless, there is an important practical consequence of our finding that c_5 is small. If c_5 vanished, then fits to the degenerate data alone would allow a determination of the *physical* B_K , since F_5 is the only *continuum* term in our

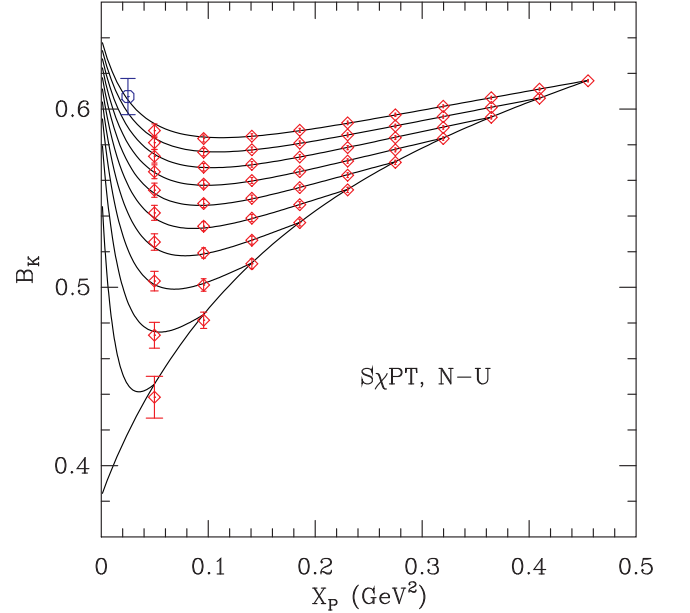


FIG. 10. $B_K(\text{NDR}, \mu = 1/a)$ as a function of X_P on the C3 ensemble. For each value of $X_P \propto m_x$, the tower of points corresponds to the different values of m_y , with m_y increasing from m_x at the bottom to $m_y^{\text{max}} = 0.05/a$ at the top. Curves show the N-U fit, both for fixed values of m_y and for $m_x = m_y$. Error bands are not included for the sake of clarity. The (blue) octagon gives $B_K(1/a)$ evaluated at the physical kaon mass with lattice artifacts removed (as described in the text).

fit function that vanishes for degenerate quark masses. Since we find c_5 to be very small, the degenerate data must play an important role in determining the extrapolated, physical B_K .

The magnitude of the parameter c_6 is expected to lie in the same range as c_4 , namely from $\Lambda_{\text{QCD}}^2 (a\Lambda_{\text{QCD}})^2$ (discretization errors) to $\Lambda_{\text{QCD}}^2 \alpha^2$ (truncation errors). Numerical values for these ranges are given in Eqs. (64) and (65) above. We see from Table XII that some of the N-U fit values of c_6 exceed the smaller of these expectations, though not the larger. As for c_7 , if discretization errors

ter assuming the observed linear dependence on quark masses, Eqs. (6)). In addition we choose physical sea-quark masses by setting $L = m_{\pi_0, \text{phys}}^2$ and $S = m_{K_0, \text{phys}}^2$. Here we are assuming Dashen’s theorem, namely that there are no electromagnetic corrections to the neutral pion and kaon masses. Other choices for tuning the quark masses lead to results that differ by much less than the statistical errors.

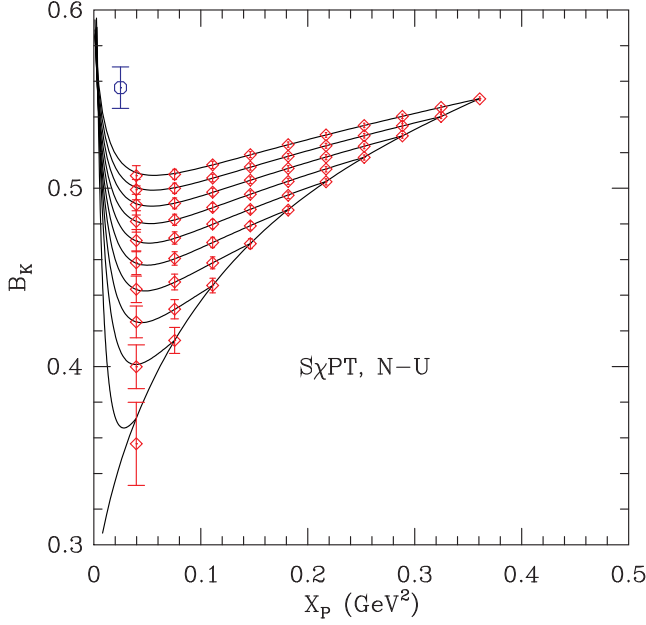


FIG. 11. $B_K(\text{NDR}, \mu = 1/a)$ as a function of X_P on the S1 ensemble. Notation as in Fig. 10.

dominate we expect

$$c_7 \approx \Lambda_{\text{QCD}}^4 (a\Lambda_{\text{QCD}})^2 \approx \begin{cases} 0.00027 \text{ GeV}^4 & (\text{coarse}) \\ 0.00014 \text{ GeV}^4 & (\text{fine}) \\ 0.000068 \text{ GeV}^4 & (\text{superfine}) \end{cases} \quad (68)$$

while if truncation errors dominate we expect

$$c_7 \approx \Lambda_{\text{QCD}}^4 \alpha_s^2 \approx \begin{cases} 0.00090 \text{ GeV}^4 & (\text{coarse}) \\ 0.00062 \text{ GeV}^4 & (\text{fine}) \\ 0.00045 \text{ GeV}^4 & (\text{superfine}) \end{cases} \quad (69)$$

In this case, some of the fit values of c_7 are larger than both of these expectations.

We have thus carried out Bayesian fits simultaneously enforcing the expected sizes of c_4 , c_6 and c_7 . As in the degenerate case, χ^2 is augmented by

$$\chi_{\text{prior}}^2 = \sum_{i=4,6,7} \frac{(c_i - a_i)^2}{\tilde{\sigma}_i^2}. \quad (70)$$

We set $a_{4,6,7} = 0$, and make two choices for the $\tilde{\sigma}_i$. The first assumes that discretization errors dominate,

$$\tilde{\sigma}_4 = \tilde{\sigma}_6 = \Lambda_{\text{QCD}}^2 (a\Lambda_{\text{QCD}})^2, \quad \tilde{\sigma}_7 = \Lambda_{\text{QCD}}^4 (a\Lambda_{\text{QCD}})^2, \quad (71)$$

while the second assumes that truncation errors dominate, and leads to the weaker constraints:

$$\tilde{\sigma}_4 = \tilde{\sigma}_6 = \Lambda_{\text{QCD}}^2 \alpha_s^2, \quad \tilde{\sigma}_7 = \Lambda_{\text{QCD}}^4 \alpha_s^2. \quad (72)$$

We label these fits N-B1 and N-B2, respectively.¹⁴

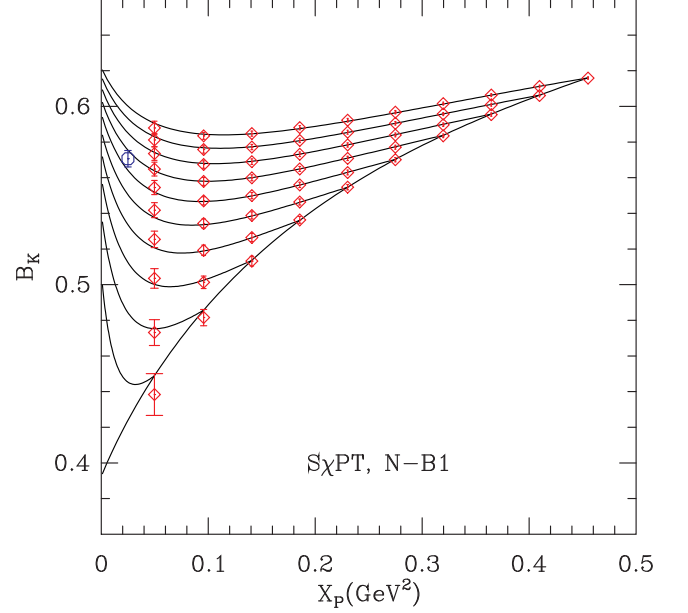


FIG. 12. $B_K(1/a)$ vs. X_P on the C3 ensemble, showing the N-B1 fit. Notation as in Fig. 10.

The N-B1 fit on the C3 ensemble is shown in Fig. 12. The fit appears to be of comparable quality to that in Fig. 10—the somewhat larger χ^2 indicates, however, that the constraints are having a non-trivial impact. The most notable changes compared to Fig. 10 are that the curvature at small X_P (and fixed m_y) is smaller, and that the value of $B_K(1/a)$ is reduced. Similar changes are seen on all ensembles.

The parameters from the N-B1 and N-B2 fits are given in Tables XIII and XIV. The values of $c_1 - c_4$ from the two fits are consistent with each other and with those from the N-U fits. For several ensembles, however, c_5 is not consistent between the fits, most notably on the C3 and C4 ensembles. This suggests that the errors are underestimated and/or that the truncated fit function does not fully represent our data. We conclude only that $|c_5| \ll 1$.

Our tentative conclusion of a non-zero c_6 from the N-U fit is not confirmed by the two new fits. Results of both signs are found. On most ensembles the results from the three fits are consistent with each other, and with a vanishing result. On C3 and C4, there are some marginal inconsistencies. We conclude that we cannot reliably determine c_6 . The situation for c_7 is slightly better, since all three fits are consistent within $2\text{-}\sigma$, and all have the same sign.

The results for $B_K(2 \text{ GeV})$ from the three fits (N-U, N-B1 and N-B2) are not consistent. The differences are

¹⁴ They correspond respectively to N-BT8 and N-BT8-2 in our previous publications.

TABLE XIII. Parameters of N-B1 fits. Notation as in Table XII.

ID	c_1	c_2	c_3	c_4	c_5	c_6	c_7	$\chi^2_{\text{aug}}/\text{dof}$	$B_K(\mu = 2\text{GeV})$
C1	0.265(30)	1.05(29)	0.06(29)	0.0026(14)	-0.033(23)	0.0021(9)	0.0002(1)	0.067(36)	0.557(12)
C2	0.312(31)	0.56(30)	0.46(30)	-0.0005(15)	-0.003(27)	0.0035(9)	0.0003(1)	0.106(53)	0.540(13)
C3	0.321(12)	0.55(12)	0.53(12)	0.0000(6)	0.005(7)	0.0017(3)	0.0004(1)	0.22(40)	0.560(5)
C4	0.309(12)	0.65(12)	0.40(12)	0.0003(6)	0.004(7)	0.0013(3)	0.0004(1)	0.283(41)	0.562(4)
C5	0.275(27)	0.94(25)	0.11(26)	0.0017(12)	-0.006(19)	0.0009(7)	0.0002(1)	0.072(39)	0.567(10)
F1	0.306(29)	0.42(24)	0.61(29)	-0.0001(8)	0.008(19)	0.0014(5)	0.0001(0)	0.103(57)	0.533(10)
F2	0.327(26)	0.24(22)	0.82(26)	-0.0006(8)	0.020(17)	0.0012(4)	0.0001(0)	0.080(51)	0.539(9)
S1	0.296(18)	0.41(16)	0.55(21)	0.0000(4)	0.011(12)	0.0005(2)	0.0001(0)	0.094(56)	0.533(7)

TABLE XIV. Parameters of N-B2 fits. Notation as in Table XII.

ID	c_1	c_2	c_3	c_4	c_5	c_6	c_7	$\chi^2_{\text{aug}}/\text{dof}$	$B_K(\mu = 2\text{GeV})$
C1	0.263(38)	1.07(37)	0.03(37)	0.0026(17)	-0.027(20)	0.0010(11)	0.0004(2)	0.021(12)	0.562(14)
C2	0.333(39)	0.35(38)	0.67(38)	-0.0014(18)	0.003(23)	0.0030(11)	0.0006(2)	0.028(15)	0.533(14)
C3	0.328(12)	0.48(12)	0.60(12)	-0.0003(6)	0.041(8)	-0.0012(9)	0.0007(1)	0.151(39)	0.585(8)
C4	0.316(12)	0.58(12)	0.48(12)	0.0000(6)	0.045(8)	-0.0020(9)	0.0007(1)	0.214(44)	0.591(8)
C5	0.277(30)	0.93(29)	0.13(29)	0.0016(13)	0.019(13)	-0.0013(11)	0.0004(2)	0.050(31)	0.585(11)
F1	0.321(38)	0.28(32)	0.77(39)	-0.0005(11)	0.011(15)	0.0010(7)	0.0003(1)	0.047(33)	0.532(8)
F2	0.340(31)	0.13(26)	0.96(31)	-0.0009(9)	0.024(11)	0.0010(7)	0.0002(1)	0.038(32)	0.539(6)
S1	0.309(23)	0.30(20)	0.71(27)	-0.0003(5)	0.038(8)	-0.0012(6)	0.0003(1)	0.036(23)	0.554(5)

most significant on the C3 and C4 ensembles, where they exceed $3\text{-}\sigma$. This is caused by the poor determination of some of the fit parameters, particularly c_5 and c_6 . Although all the fits give a reasonable representation of the data, the results obtained after extrapolation to physical quark masses and removal of taste-breaking differ. This is because c_5 and c_6 terms have significantly different dependence on quark masses and taste breaking.

To do better, we recall that, because of the smallness of c_5 , we can almost determine the *physical* B_K using the degenerate data alone. Furthermore, the D-U, D-B1 and D-B2 fits give consistent results for c_{1-4} , which we take to indicate that these results are reliable. Thus, in our last stage of SU(3) fitting, we use the results from the degenerate fits as constraints on the parameters c_{1-4} when doing fits to the full data set. We dub these “double Bayesian” fits, and think that they provide the most reliable way of determining the parameters c_5 and c_6 .

In more detail, we first do a D-B1 or D-B2 fit to degenerate points as described above. The central values and errors from these fits are then used as constraints on c_{1-4} in a second fit to the full data-set, along with the constraints on $c_{6,7}$ just described. In other words, if the first fit yields $a_i \pm \tilde{\sigma}_i$ for $i = 1, 4$, then the second fit augments the χ^2 by

$$\chi^2_{\text{prior}} = \chi^2_{\text{prior}(1)} + \chi^2_{\text{prior}(2)} \quad (73a)$$

$$\chi^2_{\text{prior}(1)} = \sum_{i=1}^4 \frac{(c_i - a_i)^2}{\tilde{\sigma}_i^2} \quad (73b)$$

$$\chi^2_{\text{prior}(2)} = \sum_{i=6,7} \frac{(c_i)^2}{\tilde{\sigma}_i^2} \quad (73c)$$

with $\tilde{\sigma}_{6,7}$ given either by Eq. (71), if the first fit was D-B1, or by (72) otherwise. We label these fits N-BB1 and N-BB2, respectively.¹⁵

Examples of the resulting fits are shown in Figs. 13 and 14, and fit parameters from all ensembles are collected in Tables XV and XVI.

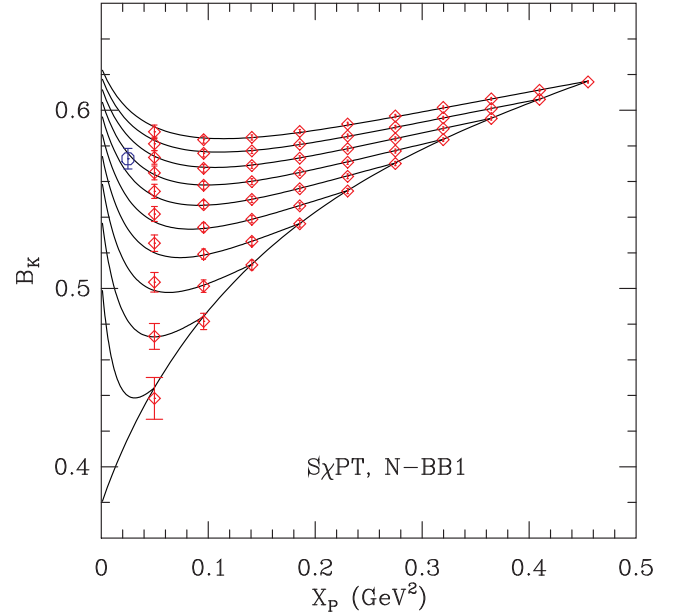


FIG. 13. $B_K(1/a)$ vs. X_P on the C3 ensemble, showing the N-BB1 fit. Notation as in Fig. 10.

¹⁵ These correspond respectively to the N-BT7 and N-BT7-2 fits in our previous publications.

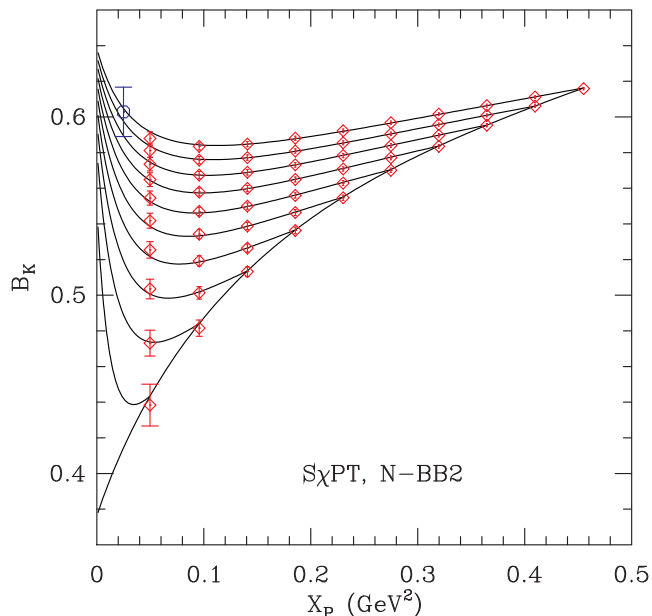


FIG. 14. $B_K(1/a)$ vs. X_P on the C3 ensemble, showing the N-BB2 fit. Notation as in Fig. 10.

The values of the c_i , the augmented χ^2 , and B_K , are consistent between fits N-B1 and N-BB1, and between N-B2 and N-BB2. The major changes induced by using two levels of constraints are that the errors in $c_1 - c_3$ are reduced, while those in B_K are increased. The former is expected, since we are placing constraints on $c_1 - c_3$ in the second stage of fitting. The latter is indicative of the uncertainties in the fit parameters induced by discretization errors, particularly in c_6 . With $c_1 - c_3$ pinned down, a greater range of c_6 values are explored, leading to more variation in the artifact subtraction needed to obtain the physical B_K . Since this variation represents a genuine systematic error in our procedure, and is best represented by the N-BB fits, we choose the N-BB fits for our central values. Specifically, we choose fit N-BB1, using N-BB2 and N-B1 to estimate different fitting systematics.

We see from tables that the difference between the physical B_K obtained from the N-BB1 and N-BB2 fits can be as large as 6% on the coarse and superfine lattices. To understand this difference in more detail, Table XVII gives a breakdown of the contributions to B_K on both the C3 and S1 ensembles. Here, valence quark masses have been set to their physical values, but taste-splittings have not been removed. In other words, this is the result of using the SChPT fit form to extrapolate the lattice data on this ensemble to the physical valence-quark masses.¹⁶ The chiral series shows poor convergence, with NLO:LO $\approx 2/3$, although the NNLO term is of reasonable size (NNLO:LO ≈ 0.13). We see also that the lattice artifacts

(given in the NLO- $c_{4,6,7}$ column) amount to no more than a quarter of the total NLO contribution on the coarse ensemble, and significantly less on the superfine ensemble. The major difference between the two N-BB1 and N-BB2 fits is in the relative size of these lattice artifacts and the continuum c_5 contribution. Since the artifacts are removed when we quote a physical value, it is this difference which is mainly responsible for the differences one sees in the physical B_K between the N-BB1 and N-BB2 fits. Given our data, it appears difficult to resolve this uncertainty.

We close this subsection by further illustrating the impact of lattice artifacts. Figures 15 and 16 show the dependence of B_K on $X_P \propto m_x$ for the N-BB1 fit on the C3 and S1 ensembles, with m_y set to the physical strange-quark mass. The three curves on each plot show the fit function itself (solid), the fit function with lattice-artifact terms removed (dashed), and the fit function with all taste-violating lattice artifacts removed, and sea-quark masses set to their physical values (dotted). The points with errors indicate the resulting “physical” $B_K(1/a)$. Only the “SU(2) regime” is shown, for which $m_x/m_s^{\text{phys}} < 1/2$. We observe that the size of the total taste-violating discretization errors (the difference between the solid and [red] dotted curves) is moderate on the course lattices (6% or less for $m_x \geq m_x^{\text{phys}}$), and somewhat reduced on the superfine lattices.

We also observe that the curvature in the fit functions is almost entirely due to the lattice-artifact terms. Removing these gives the (blue) dashed curves, which show much less curvature. The chiral logarithms that remain if lattice-artifact terms are removed are proportional to $\log(X_I)$, and thus do not diverge in the chiral limit because X_I is non-vanishing in this limit due to taste splitting. After removing taste splitting (giving the [red] dotted curves), there is a logarithmically divergent term (an SU(2) partially quenched chiral logarithm), but it has a small coefficient and is barely visible.

We summarize our experience with SU(3) fitting as follows. The fits are reasonably successful at reproducing the data, but have two major drawbacks. First, the convergence of the chiral series is suspected for physical quark masses, which undermines our use of SChPT to remove unphysical lattice artifacts. Second, our fits do not determine the size of the lattice-artifact terms, leading to significant differences between the “physical” results from different fits. To some extent these two drawbacks “cancel”, and we can hope that the systematic error we estimate by comparing fits takes into account the uncertainties in removing the artifacts. Nevertheless, the reliance on SU(3) SChPT is unsatisfactory.

E. SU(2) fitting

We now turn to our second fitting strategy, in which we fix m_y close to m_s^{phys} , extrapolate m_x to the phys-

¹⁶ We stress that the values for $B_K(1/a)$ differ from those in Figs. 13 and 14 because the latter have taste splittings set to zero.

TABLE XV. Fit parameters for N-BB1 fits. Notation is as in Table XII.

ID	c_1	c_2	c_3	c_4	c_5	c_6	c_7	$\chi^2_{\text{aug}}/\text{dof}$	$B_K(\mu = 2\text{GeV})$
C1	0.2944(54)	0.767(45)	0.386(36)	0.0015(5)	-0.017(15)	0.0023(9)	0.0002(1)	0.084(61)	0.555(12)
C2	0.3269(51)	0.409(41)	0.623(34)	-0.0011(6)	0.004(16)	0.0036(8)	0.0003(1)	0.108(54)	0.538(12)
C3	0.3345(28)	0.418(27)	0.677(20)	-0.0006(2)	0.017(5)	0.0013(5)	0.0004(1)	0.250(55)	0.562(6)
C4	0.3229(27)	0.514(26)	0.560(19)	-0.0003(2)	0.016(5)	0.0009(5)	0.0004(1)	0.320(72)	0.564(5)
C5	0.2913(51)	0.790(43)	0.285(32)	0.0011(5)	0.003(12)	0.0009(6)	0.0002(1)	0.078(61)	0.567(10)
F1	0.3243(47)	0.253(28)	0.825(35)	-0.0005(3)	0.015(12)	0.0015(5)	0.0001(0)	0.111(55)	0.535(9)
F2	0.3367(49)	0.155(32)	0.939(30)	-0.0008(3)	0.024(10)	0.0013(4)	0.0001(0)	0.078(45)	0.540(8)
S1	0.3088(33)	0.293(19)	0.727(28)	-0.0002(2)	0.016(7)	0.0006(2)	0.0001(0)	0.105(63)	0.535(6)

TABLE XVI. Fit parameters for N-BB2 fits. Notation is as in Table XII.

ID	c_1	c_2	c_3	c_4	c_5	c_6	c_7	$\chi^2_{\text{aug}}/\text{dof}$	$B_K(\mu = 2\text{GeV})$
C1	0.2801(92)	0.905(86)	0.217(67)	0.0019(7)	-0.013(15)	0.0006(17)	0.0005(2)	0.026(26)	0.564(17)
C2	0.3444(87)	0.235(82)	0.795(59)	-0.0019(7)	0.013(15)	0.0027(17)	0.0007(2)	0.029(20)	0.535(17)
C3	0.3369(27)	0.395(26)	0.700(19)	-0.0007(2)	0.056(15)	-0.0020(15)	0.0008(2)	0.164(42)	0.592(14)
C4	0.3251(26)	0.493(25)	0.580(18)	-0.0004(2)	0.060(14)	-0.0029(15)	0.0008(2)	0.227(56)	0.598(13)
C5	0.2851(68)	0.850(61)	0.213(50)	0.0013(5)	0.027(17)	-0.0017(20)	0.0004(2)	0.052(40)	0.588(19)
F1	0.3409(92)	0.112(69)	0.996(65)	-0.0010(4)	0.025(12)	0.0006(12)	0.0004(2)	0.055(31)	0.539(12)
F2	0.3511(76)	0.030(56)	1.095(50)	-0.0012(4)	0.034(13)	0.0006(14)	0.0003(2)	0.042(28)	0.545(13)
S1	0.3189(63)	0.207(46)	0.843(52)	-0.0005(2)	0.048(11)	-0.0015(10)	0.0003(1)	0.042(25)	0.560(11)

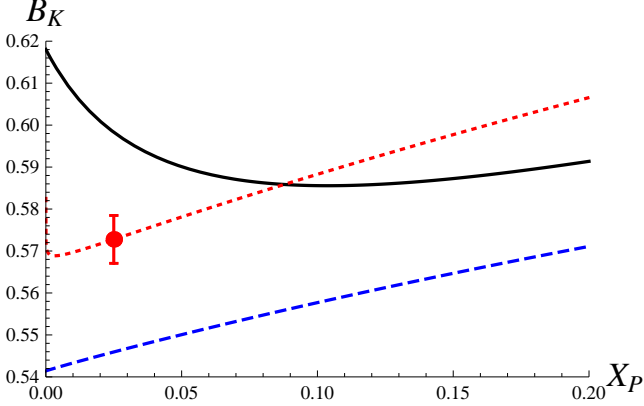


FIG. 15. $B_K(1/a)$ vs. X_P (in GeV^2) on the C3 ensemble from the N-BB1 fit. The solid (black) curve shows the fit function with $m_y = m_s^{\text{phys}}$, the dashed (blue) curve shows the result of removing the lattice-artifact terms (i.e. those proportional to c_4 , c_6 and c_7), and the dotted (red) curve shows the “physical” B_K , i.e. with lattice artifacts and taste-splittings removed, and with the sea-quarks set to their physical values. The data-point shows the result for $m_x = m_d^{\text{phys}}$.

ical light-quark mass, and then extrapolate m_y to the physical strange quark mass. At all stages we need $m_x \ll m_y \sim m_s^{\text{phys}}$ in order that SU(2) S χ PT can be used. The fitting function is given in Eqs. (41-45), and has the key feature that there are no terms arising solely from discretization or truncation errors.

On each ensemble, we extrapolate to m_s^{phys} using our three heaviest y quarks (e.g. $am_y = \{0.04, 0.045, 0.05\}$ on the coarse lattices). We label this choice “3Y”. The

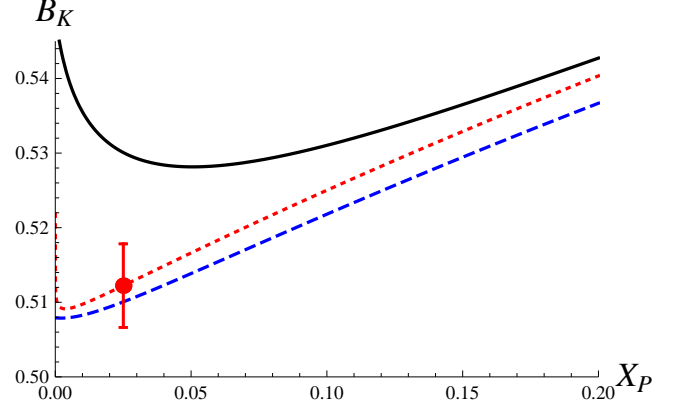


FIG. 16. As in Fig. 15 but for the S1 ensemble.

extent of the extrapolation varies substantially between the ensembles, as can be seen by comparing Tables V and VI. Our largest values of am_y lie 3%, 15% and 24% below the physical strange quark mass, on coarse, fine and superfine lattices, respectively.

For the chiral extrapolation we use either our lightest 4 or 5 values of m_x , choices we label 4X and 5X, respectively. For the “4X3Y” choice, which leads to our preferred fit, the SU(2) expansion parameter satisfies $m_x/m_y \leq 1/2$.

For fixed sea-quark masses, the fitting function (41) has three terms. The first two are present at NLO, while the third is of NNLO. We have thus tried two types of “X-fits”: NLO (two parameter) and NNLO (three parameter).

We begin by considering the 4X3Y-NLO fits. The first

TABLE XVII. Breakdown of contributions to B_K in the SChPT N-BB1 and N-BB2 fits on the C3 and S1 ensembles. Valence quark masses are set to their physical values.

Ens.	Fit	$B_K(1/a)$	LO	NLO- c_{1-2}	NLO- c_5	NLO- $c_4, c_{6,7}$	NNLO- c_3
C3	N-BB1	0.598	0.335	0.157	0.013	0.052	0.042
C3	N-BB2	0.609	0.337	0.151	0.045	0.033	0.043
S1	N-BB1	0.530	0.309	0.144	0.012	0.020	0.045
S1	N-BB2	0.529	0.319	0.125	0.038	-0.006	0.052

stage is the “X-fit”, which we do separately for the 3 largest values of am_y . The fits for the largest am_y on ensembles C3 and S1 are shown, respectively, in Figs. 17 and 18. The (mild) curvature in the fit function is due

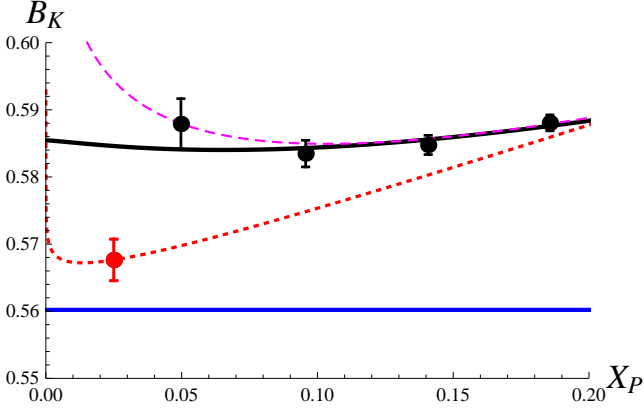


FIG. 17. $B_K(1/a)$ versus X_P on coarse ensemble C3, for $am_y = 0.05$. The 4X NLO fit is shown by the solid (black) line. The horizontal (blue) line gives the contribution of the LO term. The (magenta) dashed curve has taste-splittings removed, while the (red) dotted curve shows the result of removing taste-splittings and setting the the light sea-quark to its physical value. The (red) point on this line is at $X_P = m_{\pi_0}^2$ and shows the errors from the fit.

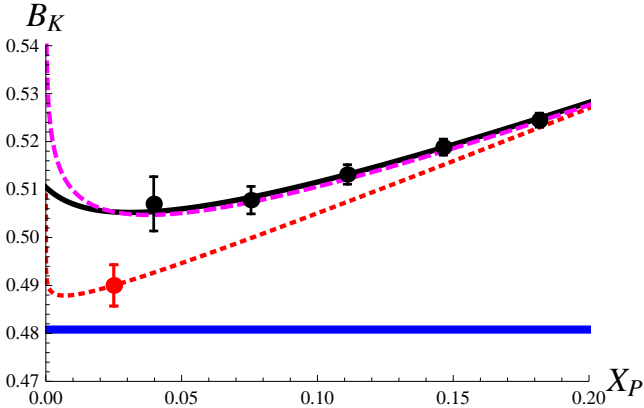


FIG. 18. $B_K(1/a)$ versus X_P on the superfine ensemble S1, for $am_y = 0.018$. The data point at the largest X_P is not included in the fit. Notation as in Fig. 17.

to the chiral logarithms. As in the SU(3) fit, these do

not diverge in the chiral limit because of taste-breaking. The figures also show the relative size of the LO and NLO contributions. The NLO contribution (difference between the solid [black] and horizontal [blue] lines) is small, not exceeding 10% over the range of the fits. The (magenta) dashed curves show the impact of removing taste-splittings from the fit function, so that the chiral logarithms do diverge. The impact is seen to be small in the region of our data, most notably on the superfine lattice. The (red) dotted curves show the form in the continuum limit, with physical light sea-quarks, as predicted by the fit. The single point on these curves shows the result (including errors from the fit) for $m_x = m_d^{\text{phys}}$. The difference between the fit and these “continuum” curves is seen mainly to be the result of setting the light sea-quark mass to its physical value, which alters the size of the chiral logarithm. This correction for unphysical effects is clearly dependent on our use of SChPT.

The solid [black] and dotted [red] curves in these figures can be compared, approximately, to the corresponding curves from the SU(3) fits shown in Figs. 15 and 16. The comparison is not exact because the values of m_y differ (most notably for the S1 ensemble). For the C3 ensemble, the curvature in the SU(2) fit is much smaller than in the SU(3) fit. This is because the curvature in the SU(3) fits is dominated by the lattice-artifact terms, which are absent in the SU(2) fits, and very poorly determined in the SU(3) fits. The SU(2) fit is superior in that it incorporates the expectation that these artifacts are small in the SU(2) regime. The continuum curves also differ between SU(2) and SU(3) fits, with the chiral log being weaker in the latter. This is possible because of $O(m_x/m_y)$ corrections to its coefficient that are of NNLO in SU(2) SChPT and thus dropped in the SU(2) fit. Again, the SU(2) fit is superior, since the size of these corrections is not known in the SU(2) limit.

The parameters of the NLO X-fits, and the resulting “physical” B_K , are given (for the largest values of am_y only) in Table XVIII for all ensembles. Some fits have a relatively high (uncorrelated) χ^2/dof , indicating that the NLO fit function may be insufficient. The fit parameters themselves are consistent on all the coarse lattices, with evidence for monotonic variation with a^2 .

After repeating this procedure for the three heaviest values of m_y , we then proceed to the “Y-fit”. The results for “continuum, physical” B_K from the X-fits are expected to be a smooth, analytic function of $Y_P \propto m_y$. Examples of the data are shown in Figs. 19 and 20.

TABLE XVIII. Fit parameters for the 4X-NLO fits. The valence strange quark mass is fixed at our heaviest value, i.e. $am_y = 0.05$, 0.03 , and 0.018 on coarse, fine and superfine lattices, respectively. The result for $B_K(1/a)$ is obtained from the fit by setting taste-splittings to zero, $L = m_{\pi_0}^2$ and $m_x = m_d^{\text{phys}}$.

ID	d_1	d_2	χ^2/dof	$B_K(\mu = 1/a)$
C1	0.548(11)	0.107(49)	0.19(26)	0.5568(96)
C2	0.556(12)	0.060(54)	0.36(37)	0.564(11)
C3	0.5602(34)	0.035(16)	0.83(54)	0.5676(31)
C4	0.5581(35)	0.039(16)	0.57(45)	0.5656(31)
C5	0.5501(85)	0.066(40)	0.08(16)	0.5582(77)
F1	0.5107(86)	0.089(45)	0.06(14)	0.5189(76)
F2	0.5146(75)	0.066(40)	0.03(11)	0.5222(66)
S1	0.4808(50)	0.143(30)	0.06(14)	0.4900(43)

We extrapolate to the “physical” value of the $\bar{s}s$ meson, $Y_P = (0.6858 \text{ GeV})^2$ [38]. In all cases, our results are consistent with a linear dependence, and so we use a fit to

$$f_{\text{ph}} = b_1 + b_2 Y_P \quad (74)$$

to obtain our central value. We also do a quadratic “fit” in order to estimate the systematic error arising from this extrapolation. The parameters from the linear 3Y

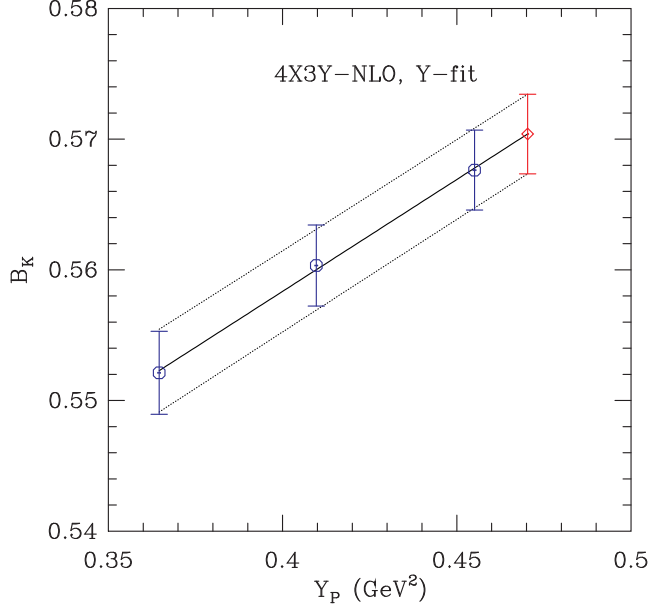


FIG. 19. $B_K(1/a)$ from the 4X-NLO fit vs. Y_P , on the coarse ensemble C3. A linear extrapolation to the physical strange quark mass is shown.

fits are collected in Table XIX. The fits are good, and the parameters are consistent across all ensembles. Also quoted are the values of B_K obtained by extrapolation in Y_P and subsequent running to the common scale 2 GeV.

We have repeated this analysis using NNLO X-fits (i.e. keeping the Q_3 term). The X-fits are shown in Figs. 21

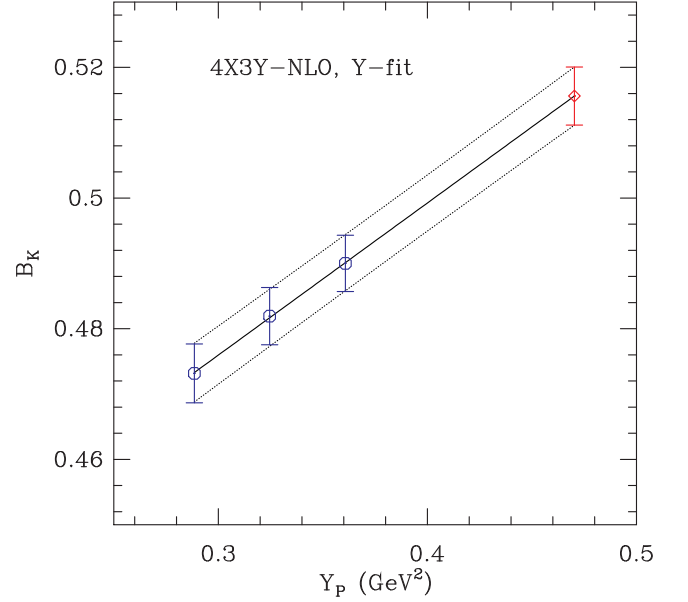


FIG. 20. $B_K(1/a)$ from the 4X-NLO fit, versus Y_P , on the superfine ensemble S1. A linear extrapolation to the physical strange quark mass is shown.

TABLE XIX. Parameters of 3Y fits (using input from 4X-NLO fits) and the resulting $B_K(\mu = 2\text{GeV})$.

ID	b_1	b_2	χ^2/dof	$B_K(\mu = 2\text{GeV})$
C1	0.482(15)	0.162(22)	0.0009(6)	0.5488(95)
C2	0.466(16)	0.214(27)	0.0026(10)	0.556(10)
C3	0.4899(47)	0.1712(73)	0.0154(26)	0.5599(30)
C4	0.4866(50)	0.1725(75)	0.0135(28)	0.5577(31)
C5	0.476(12)	0.179(19)	0.0015(8)	0.5503(75)
F1	0.433(13)	0.215(23)	0.0017(10)	0.5416(77)
F2	0.437(11)	0.214(19)	0.0037(15)	0.5454(66)
S1	0.4061(69)	0.233(14)	0.0049(17)	0.5385(46)

and 22 for the C3 and S1 ensembles, respectively. The resulting fit parameters are given in Tables XX and XXI.

TABLE XX. Fit parameters for the 4X-NNLO fits. Notation as in Table XVIII.

ID	d_1	d_2	d_3	χ^2/dof	$B_K(\mu = 1/a)$
C1	0.561(19)	-0.13(20)	0.88(60)	0.031(51)	0.565(14)
C2	0.577(21)	-0.29(23)	1.31(69)	0.090(93)	0.577(16)
C3	0.5695(60)	-0.127(67)	0.61(20)	0.16(12)	0.573(5)
C4	0.5660(62)	-0.097(68)	0.50(20)	0.12(10)	0.571(5)
C5	0.557(15)	-0.05(16)	0.42(48)	0.037(56)	0.562(11)
F1	0.517(15)	-0.04(19)	0.54(66)	0.007(25)	0.523(11)
F2	0.519(13)	-0.02(17)	0.35(57)	0.002(14)	0.525(10)
S1	0.4834(93)	+0.09(13)	0.27(51)	0.075(94)	0.491(7)

The NNLO X-fits are better able to capture the curvature seen in the data (particularly on the coarse ensembles), and have correspondingly reduced χ^2/dof . The

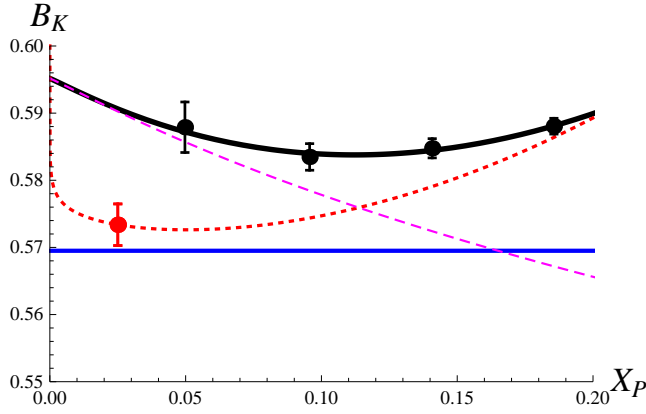


FIG. 21. $B_K(1/a)$ versus X_P on coarse ensemble C3, for $am_y = 0.05$. The 4X-NNLO fit is shown by the solid (black) line. The horizontal (blue) line gives the contribution of the LO term. The (magenta) dashed curve shows the fit with NNLO term removed. The (red) dotted curve has taste-splittings removed and the light sea-quark mass set to its physical value. The (red) point on this line is at $X_P = m_{\pi_0}^2$ and shows the errors from the fit.

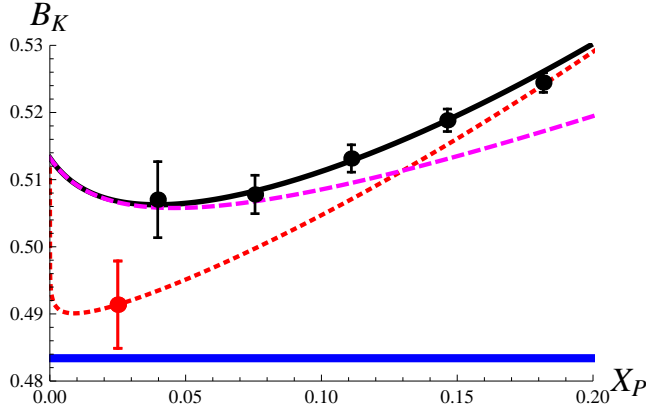


FIG. 22. $B_K(1/a)$ versus X_P on the superfine ensemble S1, for $am_y = 0.018$. Notation is as in Fig. 21. The data-point with the largest value of X_P is *not* included in the fit.

convergence of the chiral series begins to break down on the C3 ensemble at the largest fit values of X_P , in the sense that $NNLO : NLO \approx 1$. The convergence on the S1 ensembles is satisfactory for all four fit points. The linear Y-fits remain very good on all ensembles (and are not shown).

Comparing Tables XIX and XXI, we see that NLO and NNLO fits yield “physical” B_K values that are consistent within errors. We have also done NNLO 5X3Y fits, and simultaneous X and Y extrapolations (rather than sequential), both of which give consistent results.

Clearly, SU(2) SChPT fitting is much more straightforward than that to the SU(3) form. No fitting parameters need be dropped, and no Bayesian constraints are required. While we use only a fraction of our data (roughly a quarter), the points we keep are those which lie closest

TABLE XXI. Parameters of 3Y fits (using input from 4X-NNLO fits), and the resulting $B_K(\mu = 2\text{GeV})$.

ID	b_1	b_2	χ^2/dof	$B_K(\mu = 2\text{GeV})$
C1	0.501(19)	0.139(31)	0.0002(3)	0.557(14)
C2	0.478(22)	0.215(39)	0.0012(6)	0.569(16)
C3	0.5081(64)	0.144(11)	0.0053(18)	0.5651(46)
C4	0.5080(69)	0.137(11)	0.0038(17)	0.5621(47)
C5	0.493(17)	0.150(27)	0.0003(4)	0.554(11)
F1	0.443(18)	0.199(34)	0.0005(6)	0.544(11)
F2	0.444(15)	0.203(27)	0.0016(11)	0.547(10)
S1	0.4121(98)	0.220(20)	0.0020(12)	0.5385(72)

to the physical kaon and thus should dominate the extrapolation to the physical point. As for the SU(3) fits, the use of SChPT (rather than, say, an analytic extrapolation) is crucial, as it allows us to remove the effect of taste-breaking, and to account for the shift in the chiral logarithm due to the use of an unphysical light sea-quark mass. We cannot, however, claim to have provided a test of the predicted chiral logarithms; instead, we have shown consistency with our data.

We compare our SU(2) results with those from SU(3) fitting when we discuss error budgets in Sec. III H below.

F. Dependence of B_K on sea quark masses

The fits presented above are on individual ensembles, and thus at fixed values of lattice spacing and sea-quark masses. Both SU(2) and SU(3) SChPT predict, at NLO, linear dependence of B_K on the light sea-quark mass (once chiral logarithms have been accounted for, as has been done above). This enters through the H_5 term in SU(3) SChPT and the Q_4 term in SU(2) SChPT. At NLO, this dependence can either be treated as that of the leading order coefficients (b_1 and d_1) or of B_K itself. We take the latter approach, and investigate this dependence on the coarse and fine lattices, where we have results with more than one light sea-quark masses. We also discuss the impact of the mismatch of the strange sea-quark mass with its physical value.

We plot $B_K(\mu = 2\text{GeV})$ on the coarse lattices from the N-BB1 SU(3) fits against am_ℓ in Fig. 23. The data show only weak dependence on am_ℓ . We perform both constant and linear extrapolations to the physical value of am_ℓ , and find that they lead to consistent results. The corresponding results from the 4X3Y-NNLO SU(2) fit are shown in Fig. 24. Here the linear and constant fits are almost indistinguishable. We collect the results of these extrapolations in Table XXII.

For the fine lattices, we have two light sea-quark masses, corresponding approximately to $am_\ell = 0.005$ (C5) and $am_\ell = 0.01$ (C3) on the coarse lattices. As on the coarse lattices, the data have very weak dependence on am_ℓ , as shown in Figs. 25 and 26.

To quantify the size of the mass dependence, we use

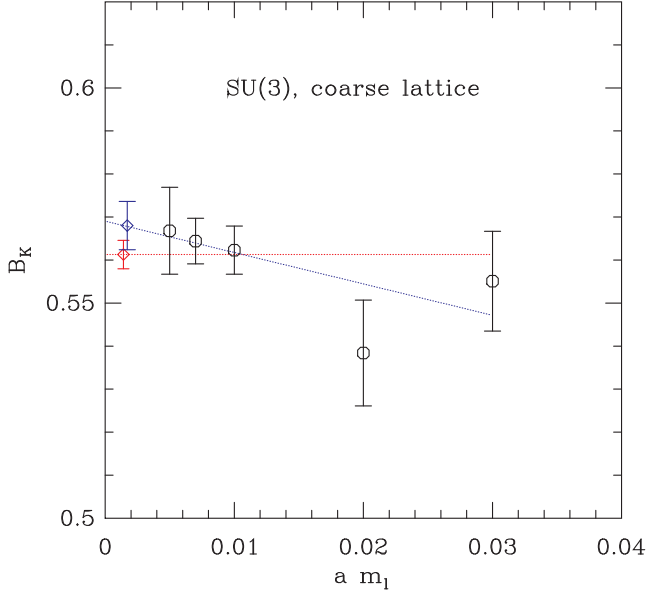


FIG. 23. $B_K(\mu = 2\text{GeV})$ from N-BB1 SU(3) fits versus am_l (light sea-quark mass) on MILC coarse ensembles. The data come from Table XV. Linear and constant fits are shown, along with the resulting value after extrapolation to the physical value of am_l . The errors on the point at $am_l = 0.01$ and 0.007 are smaller due to the larger number of measurements made on the C3 and C4 ensembles.

TABLE XXII. $B_K(\mu = 2\text{GeV})$ on the coarse lattices after extrapolation to the physical light sea-quark mass, for various choices of fit.

$S\chi\text{PT}$	valence fit	am_l fit	B_K
SU(3)	N-BB1	constant	0.5613(33)
SU(3)	N-BB1	linear	0.5680(56)
SU(2)	4X3Y-NNLO	constant	0.5628(30)
SU(2)	4X3Y-NNLO	linear	0.5624(57)

the form expected from either SU(2) or SU(3) SChPT

$$B_K(am_l) = B_K(am_l = 0) [1 - L_P/\Lambda^2] . \quad (75)$$

The linear fit shown in Fig. 23 has $\Lambda \approx 2.8$ GeV. This is much smaller than the canonical size expected in ChPT, namely $\Lambda \approx 1$ GeV, indicating that the sea-quark mass dependence is indeed weak. We note that mass dependence with this value of Λ is also consistent with the fine lattice data.

For the superfine lattices, we have only a single light sea-quark mass, and so cannot perform an extrapolation to am_l^{phys} . Instead, we assume (based on the fits above) that, to first approximation, there is no dependence on am_l . We do so consistently for all three lattice spacings, taking ensembles C3, F1 and S1 for subsequent continuum extrapolation. We use these three ensembles since they all have $m_l/m_s = 1/5$, with $m_s \approx m_s^{\text{phys}}$, and thus have closely matched sea-quark masses.

Clearly this procedure leads to a systematic error if

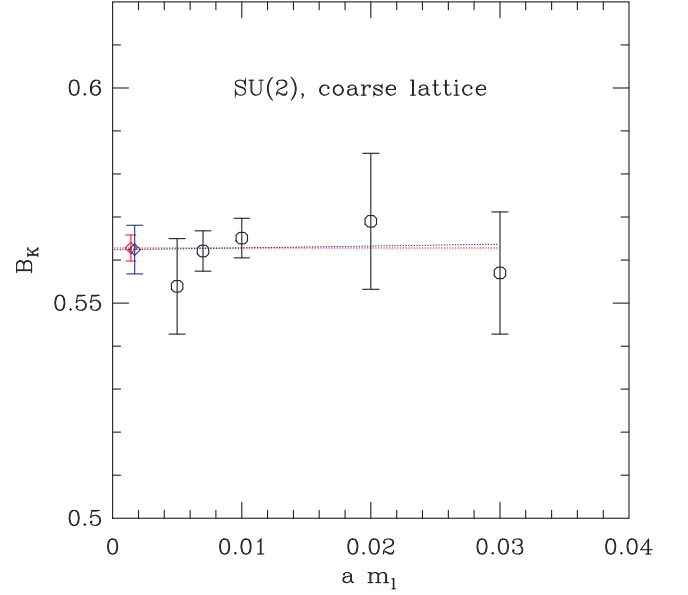


FIG. 24. $B_K(\mu = 2\text{GeV})$ from 4X3Y-NNLO SU(2) fits versus am_l (light sea-quark mass) on MILC coarse ensembles. The data come from Table XXI. Details are as in Fig. 23.

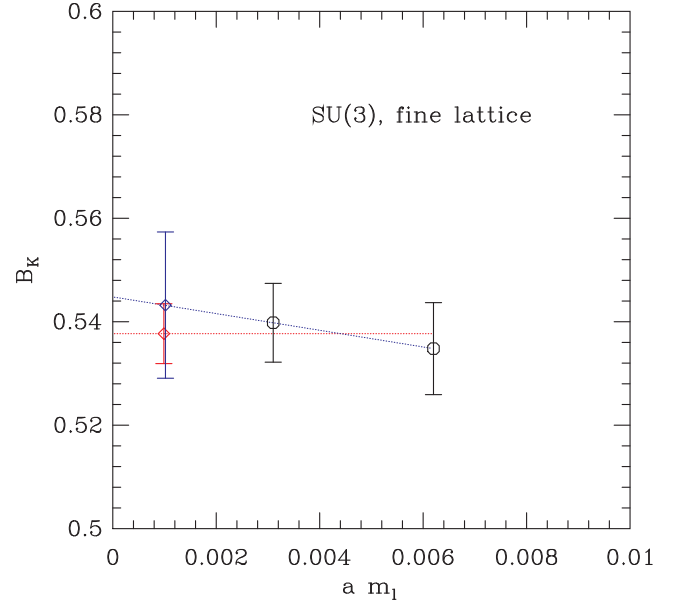


FIG. 25. As in Fig. 23 but for the MILC fine ensembles.

there is, in fact, a dependence on am_l . We estimate this error assuming that the dependence is that given by the linear fit of Fig. 23. Specifically, we take the difference between the result on ensemble C3 with that after linear extrapolation to am_l^{phys} (as given in Table XXII), divided by the C3 value, as our estimate of the fractional error from this source. The resulting error is 1.2% for the SU(3) fits, and 0.06% for the SU(2) fits. This is quite likely an underestimate in the SU(2) case, but since other sources of error are significantly larger, we have not

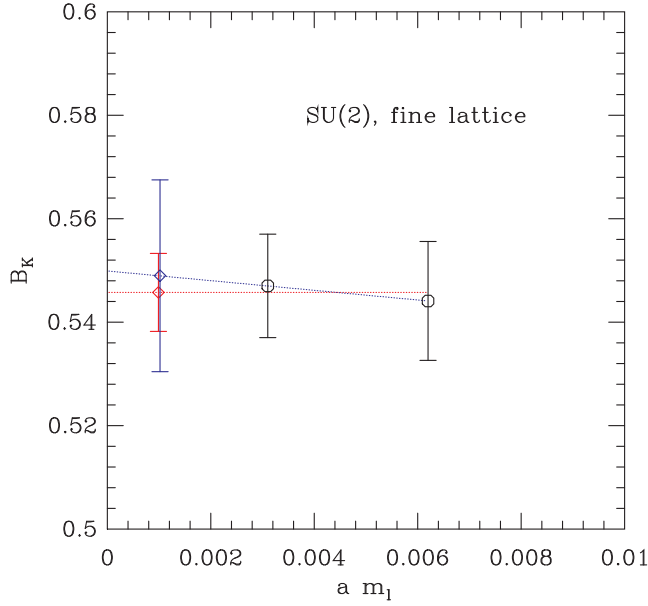


FIG. 26. As in Fig. 24 but for the MILC fine ensembles.

attempted to improve on this estimate.

TABLE XXIII. Nominal (am_s) and actual (am_s^{phys}) strange sea-quark masses [13].

MILC	am_s	am_s^{phys}
coarse	0.05	0.0350(7)
fine	0.031	0.0261(5)
superfine	0.018	0.0186(4)

Finally, we should account for the difference between the values of am_s used in the MILC ensembles and m_s^{phys} . These values are given in Table XXIII. The most reliable approach would be to have results with more than one value of am_s , and extrapolate/interpolate to the physical value. Absent this possibility, we can use SU(3) ChPT to estimate the resulting error. In SU(3) ChPT, the dependence on sea-quark masses is obtained from Eq. (75) by the substitution $L_P \rightarrow L_P + S_P/2$. Assuming this, and using the masses in Table XXIII as well as the value of Λ from the linear fit in Fig. 23, we find that the correction to $B_K(2 \text{ GeV})$ is $+0.0075$, $+0.003$, and -0.0006 for the coarse, fine and superfine ensembles, respectively. Extrapolating to the continuum linearly in a^2 leads to a final correction of -0.0028 . We take the magnitude of this correction as an estimate of the m_s mismatch error for both SU(3) and SU(2) fits.

The analysis described in this subsection does not make optimal use of all our data. For example, ensemble F2 is only used to check the (lack of) am_ℓ dependence. By doing a combined continuum and am_ℓ fit we could likely do better. We have not attempted such a fit, however, since this source of error is subdominant. Concerning the am_s mismatch, we plan in the near future to do

an exact interpolation to the physical strange sea quark mass using the reweighting technique [47].

G. Continuum extrapolation

At this stage of the analysis we have results for B_K on three lattice spacings. As noted above, we take these to be the results from the C3, F1 and S1 ensembles. Running to the common scale $\mu = 2 \text{ GeV}$ yields, for various SU(3) and SU(2) fits, the results in Tables XXIV and XXV, respectively. Most of these results have appeared in earlier Tables, but we also include results for the SU(2) 5X fits for completeness. The dominant errors remaining at this stage are those due to taste-conserving discretization and truncation errors. These depend on the lattice spacing as $a^2\alpha^n$, where $n = 0, 1, \dots$ ($n=0$ is allowed since we do not use Symanzik-improved operators), and as α^2 . Since we cannot disentangle these effects with a fit to three points, we choose to fit to a linear function of a^2 , and estimate the error we make by dropping other dependences. The resulting fits are shown in Figs. 27 and 28.

TABLE XXIV. $B_K(\mu = 2\text{GeV})$ from various SU(3) fits.

$a \text{ (fm)}$	N-U	N-B1	N-B2	N-BB1	N-BB2
0.12	0.596(10)	0.560(5)	0.585(8)	0.562(6)	0.592(14)
0.09	0.564(20)	0.533(10)	0.532(8)	0.535(9)	0.539(12)
0.06	0.581(12)	0.533(7)	0.554(5)	0.535(6)	0.560(11)

TABLE XXV. $B_K(2 \text{ GeV})$ from various SU(2) fits.

$a \text{ (fm)}$	4X3Y-NLO	4X3Y-NNLO	5X3Y-NLO	5X3Y-NNLO
0.12	0.560(3)	0.565(5)	0.558(3)	0.563(4)
0.09	0.542(8)	0.544(11)	0.541(7)	0.543(10)
0.06	0.538(5)	0.538(7)	0.539(4)	0.538(6)

TABLE XXVI. $B_K(\mu = 2\text{GeV}, a = 0)$ from various fits.

Group	fit	$B_K(\mu = 2\text{GeV}, a = 0)$
SU(3)	N-BB1	0.524(7)
SU(3)	N-BB2	0.539(14)
SU(2)	4X3Y-NLO	0.531(6)
SU(2)	4X3Y-NNLO	0.529(9)

For the SU(3) analysis, we take the continuum value obtained with N-BB1 fits as our central value. The difference between the N-BB1 and N-BB2 fits will be used as a fitting systematic, as discussed in the next subsection. To account for the fact that the discretization errors can depend on a^2 times α to some power, rather than just a^2 , we take the difference between the results on the S1 ensemble and in the continuum as the systematic error in the extrapolation. This is reasonable because including

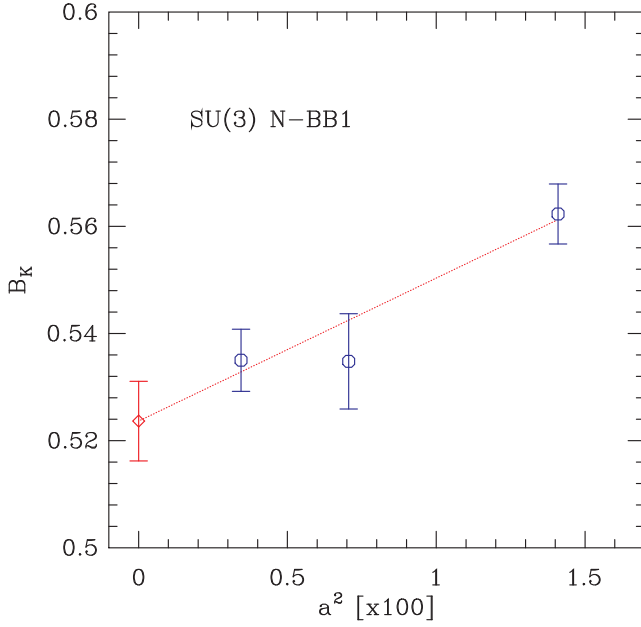


FIG. 27. $B_K(2 \text{ GeV})$ versus a^2 , together with a linear extrapolation to the continuum limit. The data is obtained using SU(3) N-BB1 fits.

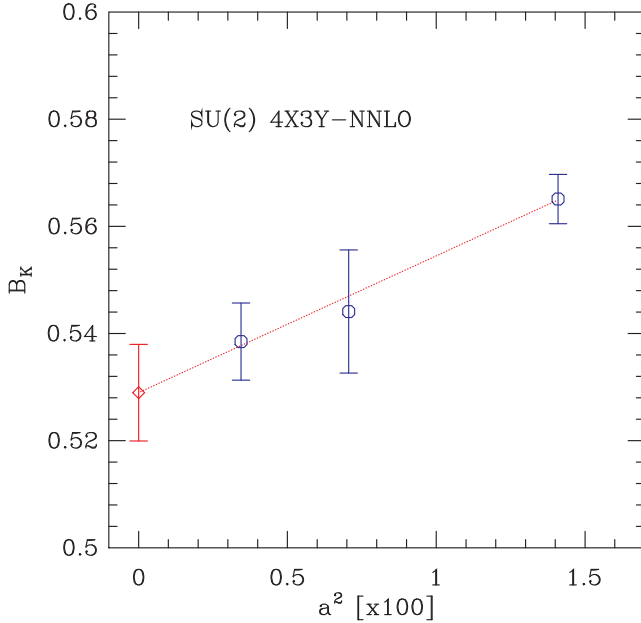


FIG. 28. $B_K(2 \text{ GeV})$ versus a^2 , together with a linear extrapolation to the continuum limit. The data is obtained using SU(2) 4X3Y-NNLO fits.

powers of α moves the S1 ensemble closer to the continuum limit relative to the other two ensembles. It is conservative because our data strongly indicates that the actual continuum value lies below that obtained on ensemble S1.

We follow the same strategy for the SU(2) analysis, using the 4X3Y-NNLO fits for our central value.

Corrections proportional to α^2 , arising from truncation of the matching factors, are estimated separately, as we now discuss.

H. Error budget

In this subsection we estimate the remaining systematic errors and combine them with those obtained above to give the complete error budget.

We begin by estimating the impact of truncating matching factors. We assume that the dominant missing term in the perturbative expansion is of magnitude $1 \times \alpha_s(1/a)^2$, and thus take as our error estimate

$$\Delta B_K^{(2)} \approx B_K^{(1)}(\mu = 1/a) \times \left[\alpha_s(\mu = 1/a) \right]^2. \quad (76)$$

Here $B_K^{(1)}$ is the one-loop corrected result, and α_s is the $\overline{\text{MS}}$ coupling constant. The resulting error estimates for the SU(3) analysis are given in Table XXVII. The results

TABLE XXVII. Estimates of the truncation error in the SU(3) analysis. We use the results of the N-BB1 fits, and ensembles C3, F1 and S1.

a (fm)	$B_K^{(1)}$	α_s	$\Delta B_K^{(2)}$
0.12	0.5728(57)	0.3286	0.062
0.09	0.5271(88)	0.2730	0.039
0.06	0.5123(56)	0.2337	0.028

for the SU(2) analysis are almost identical.

The continuum extrapolation discussed above will extrapolate away some of this error, so that the truncation error in the continuum result will be smaller than that on the superfine ensemble. To be conservative, however, we use the size of $\Delta B_K^{(2)}$ on the superfine ensemble as an estimate of this error. It is appropriate to be particularly conservative for the truncation error, since the choice of unity multiplying α^2 is a guess. We note, however, that the same approach applied to the tree-level data (i.e. using $\Delta B_K^{(1)} = B_K^{(0)} \alpha$) leads to an overestimate of the shift between $B_K^{(0)}$ and $B_K^{(1)}$ [17].

A further source of error is due to our use of a finite volume (FV). This error can be approximately accounted for by fitting to the finite volume form of the NLO SChPT result (discussed in Appendix A). We have not yet implemented this approach, however, as it is very computationally demanding. We have used SChPT to make preliminary estimates of the size of the corrections that might be induced [16]. Considering the C3 ensemble, we found that the FV error predicted by SU(3) SChPT was $\sim 1\%$ after extrapolation to physical masses. For SU(2) SChPT the error was substantially smaller. Since these two results disagree, while the actual FV error has a definite (though unknown) size, we concluded that SChPT at NLO is not a reliable quantitative guide to the size of FV effects.

Here we adopt a more direct approach. We determine the impact of changing the volume by comparing the results on ensembles C3 and C3-2. These differ only in their spatial volumes, which are 20^3 and 28^3 , respectively. The statistical weights of the two lattices are almost equal because

$$R_{\text{stat}} = \frac{20^3 \times 9 \times 671}{28^3 \times 8 \times 274} = 1.004 \dots \quad (77)$$

A comparison of results from these two lattices (after extrapolation to physical quark masses) is given in Table XXVIII. We take the difference in B_K between C3 and C3-2 ensembles as our estimate of the finite volume error. By doing so, we are effectively assuming that the 28^3 lattices have negligible FV effects, which is what is found using NLO SChPT.

TABLE XXVIII. Volume dependence of B_K . We show $B_K(\mu = 2 \text{ GeV})$ from N-BB1 fits for SU(3), and from 4X3Y-NNLO fits for SU(2).

analysis	20^3 (C3)	28^3 (C3-2)
SU(2)	0.5651(46)	0.5699(45)
SU(3)	0.5623(56)	0.5754(56)

The final source of significant error is our choice for the decay constant f_π that appears (as $1/f_\pi^2$) in the coefficient of the chiral logarithms. As should be clear from the description of our fits, our data are not sensitive enough to the contributions of chiral logarithms to allow a fit to f_π . Thus we must fix it. We choose $f_\pi = 132 \text{ MeV}$ for our central value, which is a somewhat outdated approximation to the physical value, $f_\pi = 130.4 \text{ MeV}$ [48]. (We have checked that results for B_K are indeed insensitive to the small difference between these two values.) It is reasonable to use the physical pion decay constant since the loops which give the dominant curvature are those involving pions. It is, however, equally valid at NLO to use the decay constant in the chiral limit, which is $\approx 108 \text{ MeV}$ for SU(3) ChPT and $\approx 124.2 \text{ MeV}$ for SU(2) ChPT [13]. In the SU(3) fits, it is also equally valid at NLO to use the physical value of the kaon decay constant, $f_K \approx 156.5 \text{ MeV}$ [13].

It is straightforward to determine an “ f_π error” for our SU(2) fits—we quote the difference between the results obtained with 132 MeV and 124.2 MeV. For the SU(3) fits, we have concluded after some experiments that the f_π error is already reasonably accounted for by the error we include to account for the dependence on the Bayesian fitting scheme [“fitting (2)” below]. Since we do not use the SU(3) fits for our central value, we have not pursued this further.

We now collect all sources of error and present the error budget. We do this separately for the SU(3) and SU(2) analyses. The SU(3) budget is given in Table XXIX. Most of these errors have been discussed either in this subsection or earlier ones. The exceptions are the two

“fitting” errors and the r_1 error. The first fitting error estimates the uncertainty due to the possibility of implementing the Bayesian method in different ways. The N-B1 fit uses one level of Bayesian constraints while the N-BB1 fit uses two levels. We prefer the latter method, for reasons explained earlier, but to be conservative we take the difference between the results of the two fits as a systematic error. The second fitting error estimates the impact of making different assumptions about the prior information used in the Bayesian fits. The N-BB1 fit assumes that the dominant lattice artifacts are of $\mathcal{O}(a^2)$ while the N-BB2 fit assumes that they are of $\mathcal{O}(\alpha_s^2)$. We take the difference between these fits as an estimate of this error. The resulting rather large error roughly encompasses the range of results obtained with different SU(3) fits, as can be seen from Table XXIV. Thus it can be interpreted more generally as an estimate of the error due to uncertainties in the parameters of the fit function corresponding to lattice artifacts.

TABLE XXIX. Error budget (in percent) for B_K obtained using SU(3) fitting.

cause	error (%)	memo
statistics	1.4	N-BB1 fit
matching factor	5.5	$\Delta B_K^{(2)}$ (S1)
discretization	2.2	diff. of (S1) and (a=0)
fitting (1)	0.36	diff. of N-BB1 and N-B1 (C3)
fitting (2)	5.3	diff. of N-BB1 and N-BB2 (C3)
am_l extrap	1.0	diff. of (C3) and linear extrap
am_s extrap	0.5	constant vs. linear extrap
finite volume	2.3	diff. of 20^3 (C3) and 28^3 (C3-2)
scale r_1	0.12	uncertainty in r_1

Finally, we have propagated the uncertainty in r_1 , which sets the scale, through the analysis. We do so by repeating the analysis using $r_1 \pm \sigma_{r_1}$ to determine the lattice spacing.

Our final result from the SU(3) analysis is

$$B_K(\text{NDR}, \mu = 2 \text{ GeV}) = 0.5237 \pm 0.0074 \pm 0.0438, \quad (78)$$

where the first error is statistical while the second is the sum of the systematic errors in quadrature. The former error has magnitude 1.4% and the latter 8.4%.

The error budget for the SU(2) analysis is presented in Table XXX. The errors are as for the SU(3) analysis except for the two fitting errors and the f_π error. The first of these estimates the impact on the fit of the addition of higher-order terms in the SChPT fit function used for the X-fits. Specifically, we use the difference between the NLO and NNLO fits on the 4X3Y data set. The second fitting error results from the uncertainty in the Y-fit: we quote the difference between linear and quadratic extrapolations to the physical point $Y_P = 2m_K^2 - m_\pi^2$.

Our final result of B_K from the SU(2) analysis is

$$B_K(\text{NDR}, \mu = 2 \text{ GeV}) = 0.5290 \pm 0.0090 \pm 0.0316, \quad (79)$$

TABLE XXX. Error budget of B_K in the SU(2) fitting.

cause	error (%)	memo
statistics	1.7	4X3Y NNLO fit
matching factor	5.5	$\Delta B_K^{(2)}$ (S1)
discretization	1.8	diff. of (S1) and (a=0)
fitting (1)	0.92	X-fit (C3)
fitting (2)	0.08	Y-fit (C3)
am_l extrap	0.48	diff. of (C3) and linear extrap
am_s extrap	0.5	constant vs. linear extrap
finite volume	0.85	diff. of 20^3 (C3) and 28^3 (C3-2)
r_1	0.14	r_1 error propagation
f_π	0.38	132 MeV vs. 124.4 MeV

with the statistical error having magnitude 1.7% and the total systematic error 5.9%. Although the statistical error in this result is somewhat larger than that from the SU(3) analysis—a reflection of the smaller data set used in the chiral extrapolation—the systematic error is considerably smaller, as is the total error. This improvement results from the simplifications which occur in SU(2) SchPT, which allow more straightforward fitting. Because of this we use the SU(2) analysis for our final result, while the SU(3) result provides a consistency check.

IV. CONCLUSION

We have presented a calculation of B_K using improved staggered fermions with 2+1 flavors of dynamical fermions. We have used three different lattice spacings and multiple choices for the valence and sea quark masses in order to perform the extrapolations to the continuum limit and the physical quark masses. A key ingredient in our analyses are the fitting forms predicted by staggered chiral perturbation theory, for they allows us to determine, and remove, the impact of taste-breaking discretization errors. To carry out our analysis, we needed to generalize previous SU(3) SchPT results for B_K by including the effects of a mixed action (HYP-smear valence quarks on asqtad sea) and by extending the results to SU(2) SchPT.

A striking feature of the SU(2) SchPT result is that taste-violating discretization and truncation errors do not enter until NNLO. This means that fitting to the NLO form (as we do) is as simple as for fermions with exact chiral symmetry. Although we can use only a quarter of our complete data set when doing the SU(2) fits, this is more than compensated for by the simplicity of fitting.

Using these ChPT results, we have performed two independent analyses based, respectively, on SU(3) and SU(2) SchPT at NLO. The former requires some ad-hoc simplifications of the fitting expression and the use of Bayesian constraints to obtain stable fits. We find significant sensitivity to the prior information used in the

Bayesian procedure, as well as to the precise implementation. The SU(2) fitting does not require the use of Bayesian priors, and leads to significantly smaller overall errors. Hence, we use the SU(2) analysis for our final result (79), while the SU(3) result, (78), provides a consistency check. Converting to the renormalization group invariant B_K , we find

$$\hat{B}_K = 0.7243 \pm 0.0123 \pm 0.0433 = 0.724 \pm 0.045. \quad (80)$$

Our result is in good agreement with the two existing results which use 2+1 flavors of dynamical fermions and control all sources of error. Aubin, Laiho and Van de Water use a mixed action with valence domain wall fermions on the MILC coarse and fine lattices and find [3]

$$\hat{B}_K = 0.724 \pm 0.008 \pm 0.028 = 0.724 \pm 0.029., \quad (81)$$

The RBC-UKQCD collaboration use both domain wall valence and sea quarks, finding [21]

$$\hat{B}_K = 0.720 \pm 0.013 \pm 0.037 = 0.724 \pm 0.039 \quad (82)$$

on a single lattice spacing of 1.16 fm, with a preliminary result (without an estimate of finite volume effects)

$$\hat{B}_K = 0.738 \pm 0.026 \quad (83)$$

based on two lattice spacings ($a = 0.86$ and 1.16 fm) and updated fitting [5]. Our result is also consistent with the previous calculation using staggered fermions on the coarse ($a \approx 0.12$ fm) MILC lattices, which finds $\hat{B}_K = 0.83 \pm 0.02 \pm 0.18$ [8]. We view it as a significant success of lattice QCD that results with valence staggered and domain-wall fermions are consistent. Although our calculation shares lattices with that of Ref. [3], it is completely independent of that of Refs. [5, 21].

Compared to the other two calculations, our result has the advantages of pushing closer to the continuum limit, and being based on three, rather than two lattice spacings. It also is significantly computationally cheaper. A clear disadvantage is that we must deal with taste-breaking, although this is much less of an issue with SU(2) fitting than with SU(3). Even with SU(2) fitting, however, we do rely on ChPT to a greater extent than the other calculations.

Our total error (6%) is significantly larger than that from the other calculations—4.1% for Ref. [3] and 3.5% for Ref. [5]. This difference is due almost entirely to our use of perturbative (one-loop) matching rather than the non-perturbative renormalization (NPR) used by the other calculations. Quantitatively, our matching error is 5.5% to be compared to 3.3% [3] and 2.4% [5]. In addition, our error estimate is less reliable since it is based on an educated guess about the possible size of two-loop terms.

It is also of interest to compare our result to those obtained using two flavors of dynamical quarks (but a quenched strange quark). This may be a useful comparison because there are indications that the dependence of

dimensionless quantities on the mass of the strange sea quark is weak. Indeed, our results for B_K show weak dependence on all sea-quark masses, at least within the range of masses that we consider. In fact, our result is consistent with those obtained from $N_f = 2$ calculations. A recent calculation using twisted-mass fermions, and with all errors controlled except for that due to quenching the strange quark, finds $\hat{B}_K = 0.729 \pm 0.025 \pm 0.016$ [49]. The result obtained with dynamical overlap fermions, using a single lattice spacing, is $\hat{B}_K = 0.758 \pm 0.006 \pm 0.071$ [50]. With domain-wall fermions on a single lattice spacing, the result is $\hat{B}_K = 0.699 \pm 0.025$ [51]. Finally, with improved (untwisted) Wilson fermions, the best result is $\hat{B}_K = 0.69 \pm 0.18$ [52].

Looking into the future, our result would be competitive with those of the other $N_f = 2 + 1$ calculations if we could reduce our matching error down to the level achieved in these other calculations. We are following two approaches to achieve this reduction: using 2-loop perturbation theory (which would reduce our error estimate to 1.3%, although this would remain an educated guess) and using NPR to determine the matching factors (which we expect to lead to similar errors to those quoted by the other calculations.) First results with NPR using improved staggered fermions and studying bilinear matching factors are encouraging [53].

Other improvements that are underway are the use of a fourth lattice spacing (the MILC “ultrafine” lattices with spacing $a \approx 0.045$ fm), SU(2) fitting including the finite-volume effects predicted by NLO SchPT, simultaneous continuum and chiral fitting, the addition of ensembles with other values of the strange quark mass, and an improvement in statistics for all ensembles. We are also calculating matrix elements of operators with other Dirac structures, since these are needed to constrain some theories of physics beyond the standard model.

ACKNOWLEDGMENTS

C. Jung is supported by the US DOE under contract DE-AC02-98CH10886. The research of W. Lee is supported by the Creative Research Initiatives program (3348-20090015) of the NRF grant funded by the Korean government (MEST). The work of S. Sharpe is supported in part by the US DOE grant no. DE-FG02-96ER40956. Computations for this work were carried out in part on QCDOC computers of the USQCD Collaboration at Brookhaven National Laboratory. The USQCD Collaboration are funded by the Office of Science of the U.S. Department of Energy.

Appendix A: Functions appearing in SU(3) fitting

In this appendix we present explicit forms for the functions that appear in the NLO SU(3) SchPT result for B_K with a mixed action.

1. $\mathcal{M}_{\text{conn}}^0$ and $\mathcal{M}_{\text{disc}}^0$

These two functions are unaffected by the use of a mixed action, and thus can be obtained from the general results of Ref. [18]. Here we give fully explicit results for $2 + 1$ flavors. First we have

$$\begin{aligned} \mathcal{M}_{\text{conn}}^0 &= \sum_B \tau^B F_B^{(3)} \\ F_B^{(3)} &= -\frac{1}{2} \left[(G + X_B) \ell(X_B) + (G + Y_B) \ell(Y_B) \right. \\ &\quad \left. + 2(G - K_B) \ell(K_B) - 2G K_B \tilde{\ell}(K_B) \right], \end{aligned} \quad (\text{A1})$$

where the coefficients give the relative weight of the different tastes in the loop

$$\begin{aligned} \tau^I &= 1/16, & \tau^P &= 1/16, & \tau^V &= 4/16, \\ \tau^A &= 4/16, & \tau^T &= 6/16, \end{aligned} \quad (\text{A2})$$

which are simply proportional to their multiplicity. Two standard chiral logarithmic functions arise:

$$\ell(X) = X \left[\log(X/\mu_{\text{DR}}^2) + \delta_1^{\text{FV}}(X) \right], \quad (\text{A3})$$

$$\begin{aligned} \tilde{\ell}(X) &= -\frac{d\ell(X)}{dX} \\ &= -\log(X/\mu_{\text{DR}}^2) - 1 + \delta_3^{\text{FV}}(X), \end{aligned} \quad (\text{A4})$$

where μ_{DR} is the scale introduced by dimensional regularization. The μ_{DR} dependence of the chiral logarithms is canceled by a corresponding dependence of the LECs. When we evaluate $\ell(X)$ and $\tilde{\ell}(X)$ numerically we set $\mu_{\text{DR}} = 0.77$ GeV. Although we do not use them in this paper, we quote for completeness the finite volume corrections to the chiral logarithms: by

$$\delta_1^{\text{FV}}(M^2) = \frac{4}{ML} \sum_{\vec{n} \neq 0} \frac{K_1(|\vec{n}|ML)}{|\vec{n}|} \quad (\text{A5})$$

$$\delta_3^{\text{FV}}(M^2) = 2 \sum_{\vec{n} \neq 0} K_0(|\vec{n}|ML), \quad (\text{A6})$$

with L the box size, and \vec{n} is a vector labeling the image position. These formulae assume that the time direction is much longer than the spatial directions—but this is easily corrected by including images in the time direction as well. The image sums converge fairly quickly because the Bessel functions die off exponentially. For further discussion of the properties of these functions see Refs. [54] and [19].

The second function is given by

$$\begin{aligned}
6\mathcal{M}_{\text{disc}}^0 = & \ell(\eta_I)(G + \eta_I) \frac{(X_I - Y_I)^2 (L_I - \eta_I)(S_I - \eta_I)}{(X_I - \eta_I)^2 (Y_I - \eta_I)^2} \\
& + \frac{(L_I - X_I)(S_I - X_I)}{(\eta_I - X_I)} \left[\tilde{\ell}(X_I)(G + X_I) - \ell(X_I) + \ell(X_I)(G + X_I) \left(\frac{1}{L_I - X_I} + \frac{1}{S_I - X_I} - \frac{1}{\eta_I - X_I} - \frac{2}{Y_I - X_I} \right) \right] \\
& + \frac{(L_I - Y_I)(S_I - Y_I)}{(\eta_I - Y_I)} \left[\tilde{\ell}(Y_I)(G + Y_I) - \ell(Y_I) + \ell(Y_I)(G + Y_I) \left(\frac{1}{L_I - Y_I} + \frac{1}{S_I - Y_I} - \frac{1}{\eta_I - Y_I} - \frac{2}{X_I - Y_I} \right) \right]. \quad (\text{A8})
\end{aligned}$$

As is clear from this form, this is the taste singlet contribution to the “disconnected” matrix element, i.e. that involving hairpin vertices in the loop. The poles in this function are misleading: due to cancellations between terms, the function vanishes when $X_I \rightarrow Y_I$ and is finite when X_I or Y_I equal L_I , S_I or η_I . These cancellations could lead to numerical instability in the evaluation of $\mathcal{M}_{\text{disc}}$, but we have not found this to be the case. Finally, we note that $\mathcal{M}_{\text{disc}}^0$ is independent of the renormalization scale μ_{DR} .

2. Definitions of $F_{\text{B}}^{(4)}$, $F^{(5)}$ and $F^{(6)}$

These three functions describe the contributions to $\mathcal{M}_{\text{conn}}$ that result from discretization and truncation errors. Their origin is explained in detail in Ref. [18]. Their forms are

$$F_{\text{B}}^{(4)} = \frac{3}{8f_{\pi}^2 G} \left[2G \tilde{\ell}(K_{\text{B}}) + (\ell(X_{\text{B}}) + \ell(Y_{\text{B}}) - 2\ell(K_{\text{B}})) \right] \quad (\text{A9})$$

$$F^{(5)} = \frac{3}{8f_{\pi}^2 G} [(\ell(X_{\text{V}}) + \ell(Y_{\text{V}}) - 2\ell(K_{\text{V}})) + (\ell(X_{\text{A}}) + \ell(Y_{\text{A}}) - 2\ell(K_{\text{A}}))] \quad (\text{A10})$$

$$F^{(6)} = \frac{3}{8f_{\pi}^2 G} [\ell(X_{\text{T}}) + \ell(Y_{\text{T}}) - 2\ell(K_{\text{T}})] \quad (\text{A11})$$

Note that $F^{(5)}$ and $F^{(6)}$ vanish when $m_x = m_y$.

For completeness, we give the relationships between the coefficients of these functions and those appearing in

the operator enumeration used in Ref. [18]. The coefficients of the functions $F_{\text{B}}^{(4)}$ are

$$b_j = \sum_{\text{B}'} \frac{1}{64\pi^2 f^4} \left(C_{\chi}^{1\text{B}'} g^{\text{B}'\text{B}} - C_{\chi}^{2\text{B}'} h^{\text{B}'\text{B}} \right) \quad (\text{A12})$$

where B' runs over all five tastes, and $\text{B}=\text{I}, \text{P}, \text{V}, \text{A}$ and T , for $j = 6, 7, 8, 9$ and 10 , respectively. The matrices g and h are defined in Ref. [18], and f is the chiral limit value of the pion decay constant. The coefficient of $F^{(5)}$ is

$$b_{11} = -\frac{4}{\pi^2 f^4} (C_{\chi}^{2\text{P}} + 6C_{\chi}^{2\text{T}}), \quad (\text{A13})$$

while that of $F^{(6)}$ is

$$b_{12} = -\frac{48}{\pi^2 f^4} C_{\chi}^{2\text{T}}. \quad (\text{A14})$$

We can use these formulae to estimate the order of magnitude of the coefficients b_{6-12} . One expects that $C_{\chi} \sim a^2 \Lambda_{\text{QCD}}^8$ and/or $\sim a^2 \Lambda_{\text{QCD}}^6$, while the numerical factors vary from $g^{\text{B}'\text{B}}/(64\pi^2) \sim 1/10$ to $48/\pi^2 \sim 5$. We simply treat them as of $O(1)$, and $f \sim \Lambda_{\text{QCD}}$, leading to the crude estimate given in Eq. (35).

3. Definitions of $F_{\text{B}}^{(1)}$ and $F_{\text{B}}^{(2)}$

These functions give the form of the contributions to $\mathcal{M}_{\text{disc}}$ that arise from discretization and truncation errors. The explicit 2+1 flavor form of $F_{\text{B}}^{(1)}$ can be deduced from the general result given in Ref. [18]:

$$\begin{aligned}
F_B^{(1)} = \frac{3}{8f_\pi^2 G} & \left[\ell(\eta_B) \frac{(Y_B - X_B)^2 (L_B - \eta_B)(S_B - \eta_B)}{(X_B - \eta_B)^2 (Y_B - \eta_B)^2 (\eta'_B - \eta_B)} \right. \\
& + \ell(\eta'_B) \frac{(Y_B - X_B)^2 (L_B - \eta'_B)(S_B - \eta'_B)}{(X_B - \eta'_B)^2 (Y_B - \eta'_B)^2 (\eta_B - \eta'_B)} \\
& + \frac{(L_B - X_B)(S_B - X_B)}{(\eta_B - X_B)(\eta'_B - X_B)} \left\{ \tilde{\ell}(X_B) + \ell(X_B) \left(\frac{1}{L_B - X_B} + \frac{1}{S_B - X_B} \right. \right. \\
& \quad \left. \left. - \frac{2}{Y_B - X_B} - \frac{1}{\eta_B - X_B} - \frac{1}{\eta'_B - X_B} \right) \right\} \\
& + \frac{(L_B - Y_B)(S_B - Y_B)}{(\eta_B - Y_B)(\eta'_B - Y_B)} \left\{ \tilde{\ell}(Y_B) + \ell(Y_B) \left(\frac{1}{L_B - Y_B} + \frac{1}{S_B - Y_B} \right. \right. \\
& \quad \left. \left. - \frac{2}{X_B - Y_B} - \frac{1}{\eta_B - Y_B} - \frac{1}{\eta'_B - Y_B} \right) \right\} \left. \right]. \quad (A15)
\end{aligned}$$

The coefficients of this function are

$$b_j = \frac{1}{\pi^2 f^4} (2C_\chi^{2B} + C_\chi^{3B}) a^2 \delta_B^{\text{MA1}}, \quad (A16)$$

with B=V and A for $j = 13$ and 14 , respectively.

The function $F_B^{(2)}$ occurs only in the mixed-action set-up. Its form is

$$\begin{aligned}
F_B^{(2)} = \frac{3}{8f_\pi^2 G} & \left[2 \frac{\ell(Y_B) - \ell(X_B)}{Y_B - X_B} \right. \\
& \left. + \tilde{\ell}(X_B) + \tilde{\ell}(Y_B) \right]. \quad (A17)
\end{aligned}$$

with the coefficients being

$$b_j = \frac{1}{\pi^2 f^4} (2C_\chi^{2B} + C_\chi^{3B}) a^2 \delta_B^{\text{MA2}}, \quad (A18)$$

with B=V and A for $j = 15$ and 16 , respectively.

The functions $F_B^{(1,2)}$ are finite—the apparent poles have vanishing residues—and vanish when $m_x = m_y$. They are also independent of the renormalization scale.

The size of the coefficients b_{13-16} can be estimated as for b_{6-12} , with the result given in Eq. (40).

Appendix B: Functions appearing in SU(2) SchPT fitting

In this appendix we describe how SU(2) SchPT results are obtained from SU(3) SchPT results using the recipe outlined in Sec. II C and justified in Appendix C. The recipe is to first expand the SU(3) result in powers of m_x/m_s and m_ℓ/m_s (where m_s is here meant generically, and thus includes m_y), keeping only terms of the desired order (here NLO) in SU(2) power-counting. Secondly, one allows the LECs which are constants in SU(3) SchPT to become arbitrary functions of m_s and m_y , with the exception of f .

First consider the analytic terms H_2-H_5 of Eqs. (24)–(27). H_2 is proportional to $(m_x + m_y)/\Lambda_{\text{QCD}}$. The m_x/Λ_{QCD} part remains in SU(2) ChPT, while the m_y/Λ_{QCD} part is absorbed into the m_s dependence of the overall constant B_0 . Similarly, $H_3 \propto (m_x + m_y)^2$ leads to an analytic term proportional to m_x^2 , to an m_s dependence of the analytic term linear in m_x , and to an m_s^2 dependence of B_0 . $H_4 \propto (m_x - m_y)^2/(m_x + m_y)$ leads to analytic terms of all orders in m_x , with the expansion parameter being m_x/m_y rather than m_x/Λ_{QCD} . Finally, $H_5 \propto (2m_\ell + m_s)$ leads to an analytic term linear in m_ℓ and further m_s dependence of B_0 . The net result is that given in Eqs. (41-45), with the coefficients d_j having an arbitrary dependence on the strange quark mass(es).

Next consider the continuum-like chiral logarithms in H_1 [see Eq. (22)]. Here a key role is played by the overall multiplier $1/G = 1/m_{xy;\text{P}}^2$, which arises from the denominator in the definition of B_K . In order for a term from $\mathcal{M}_{\text{conn}}^0$ or $\mathcal{M}_{\text{disc}}^0$ to remain of NLO in the SU(2) limit, it must come with a compensating factor of G , m_y or m_s . If not, it will become of NNLO in SU(2) SchPT.

To see how this works in more detail, consider first $\mathcal{M}_{\text{conn}}$, given in Eq. (A1). Its contribution to H_1 consists of a sum over tastes B of the expression

$$\begin{aligned}
\frac{F_B^{(3)}}{8\pi^2 f_\pi^2 G} = & -\frac{1}{2} \frac{1}{8\pi^2 f_\pi^2} \left[\left(1 + \frac{X_B}{G} \right) \ell(X_B) + \right. \\
& \left. \left(1 + \frac{Y_B}{G} \right) \ell(Y_B) + 2 \left(1 - \frac{K_B}{G} \right) \ell(K_B) - 2K_B \tilde{\ell}(K_B) \right]. \quad (B1)
\end{aligned}$$

On the first line, the term proportional to $\ell(X_B)$ survives as a chiral log in the SU(2) limit, but its coefficient simplifies to 1. This is because $X_B/G \propto m_x/m_s$ (where “ m_x ” can include taste-breaking $O(a^2)$ corrections) is suppressed in the SU(2) limit, so that, given the overall factor of $1/(8\pi^2 f_\pi^2)$, the X_B/G term is of NNLO in SU(2) power counting. On the second line, no terms give rise to SU(2) chiral logarithms, because the argument of the logarithms is finite when either m_x or m_ℓ vanishes. Thus one can expand the logarithms about $m_x = 0$, leading to

a power series in m_x , in which subsequent terms are suppressed by m_x/m_y or $m_x/\Lambda \propto X_P/\Lambda_\chi^2$, with $\Lambda = \Lambda_{\text{QCD}}$. The same is true for the factors multiplying the logarithms. For example, the last term on the second line becomes

$$\frac{K_B \tilde{\ell}(K_B)}{8\pi^2 f_\pi^2} \propto \frac{K_B \log[K_B]}{\Lambda_\chi^2} \quad (\text{B2})$$

$$\propto \frac{(m_x + m_y + a^2 \Lambda^3)}{\Lambda} \log[m_x + m_y + a^2 \Lambda^2] \quad (\text{B3})$$

$$\propto f_1(m_y/\Lambda) + f_2(m_y/\Lambda) \frac{m_x}{\Lambda} + \dots \quad (\text{B4})$$

The analytic dependence on m_y in f_1 can then be absorbed into the (unknown) m_y dependence of the overall coefficient multiplying H_1 [which becomes d_1 in the SU(2) SchPT result, Eq. (41)]. The dependence on m_x , which is of NLO, can be absorbed into the coefficient of the analytic term Q_3 . In this way, one finds that the entire second line can be absorbed into the coefficients d_1 and d_2 . The overall result is that

$$\frac{\mathcal{M}_{\text{conn}}^0}{G} \xrightarrow{\text{SU}(2)} -\frac{1}{2} \sum_B \tau^B \ell(X_B). \quad (\text{B5})$$

Turning next to the contribution of $\mathcal{M}_{\text{disc}}^0$, Eq. (A8), we find two new features. First, there are logarithms (and other functions) of η_I . But since $\eta_I = L_I/3 + 3S_I/3$ does not vanish when m_ℓ and a^2 vanish, these functions can be expanded just as logarithms of Y_B and K_B were above. Second, we need to use the result¹⁷

$$\frac{(S_I - X_I)}{(\eta_I - X_I)} \xrightarrow{\text{SU}(2)} \frac{3}{2}. \quad (\text{B6})$$

Third, one can have contributions proportional to $(m_\ell/\Lambda)f(m_s/\Lambda)$ (e.g. from the $L_I \tilde{\ell}(Y_I)G$ term) which can be absorbed into the coefficient d_4 .

Given these results, combined with the arguments used above for $\mathcal{M}_{\text{conn}}^0$, we find that the only terms in $\mathcal{M}_{\text{disc}}^0$ which cannot be absorbed into the coefficients d_1 , d_2 and d_4 , are those multiplying $\ln X_I$. Of these, we must pick out those multiplied by G , since all others are suppressed by m_x/m_s and thus become of NNLO. The result is then

the greatly simplified form

$$\frac{6\mathcal{M}_{\text{disc}}^0}{G} \xrightarrow{\text{SU}(2)} \frac{3}{2} \times \left[(L_I - X_I) \tilde{\ell}(X_I) + \ell(X_I) \right. \\ \left. + \text{analytic} + \text{NNLO} \right]. \quad (\text{B7})$$

All the remaining functions which arise in SU(3) SchPT, H_6 - H_{16} , become of NNLO in the SU(2) limit. This is because all the functions which appear, $F^{(2)} - F^{(6)}$, contain a factor of $1/G$ from the definition of B_K , but no counterbalancing factor in the numerator *when the numerator contains a logarithm of X_B* . For example, in $F_B^{(4)}$ [see Eq. (A9)] part of the numerator contains a factor of G , but this multiplies $\ln K_B$, which gives only terms analytic in m_x . The part containing $\ln X_B$ is multiplied by X_B , leading to the additional suppression by $X_B/G \sim m_x/m_s$.

The final result, given in Eq. (41), therefore contains no LECs induced by discretization or truncation errors.

Appendix C: Derivation of recipe for obtaining the SU(2) SchPT result

In this Appendix we demonstrate the validity of the recipe used to obtain the SU(2) SchPT result. This recipe has been described in Sec. II C and implemented in Appendix B. The essential claim is that it is sufficient to take the SU(2) limit of the NLO SU(3) expression, as long as one allows LECs (except for f) to have an arbitrary dependence on m_s and m_y . This arbitrary dependence is the only impact of working to all orders in $r_s = m_s/\Lambda_{\text{QCD}}$.¹⁸ In other words, higher orders in SU(3) SchPT do not lead to new types of functional dependence on the small quantities X_P/Λ_χ^2 , L_P/Λ_χ^2 , $a^2 \Lambda_{\text{QCD}}^2$ and α^2 .

In particular, looking at the final result of Eqs. (41-45), there are two main features in need of justification:

1. There are no terms involving those LECs that are proportional to a^2 or α^2 .
2. The chiral logarithm in the function Q_1 [Eq. (42)] is fully predicted. In particular, its coefficient ($\propto 1/f^2$) has no dependence on m_s .

Discretization and truncation errors thus enter only through the coefficient d_1 [see Eq. (46)] and through the masses of the PGBs in the chiral logarithms.

The approach we use to justify the recipe is to use SU(3) ChPT, working to all orders in r_s , but only at NLO in light-quark masses, discretization and truncation effects. This amounts to a “poor-person’s SU(2) SchPT”,

¹⁷ One might be concerned about the legitimacy of the SU(2) limit, since ChPT does not control the m_s dependence when m_s becomes large. If one looks back at the source of this ratio in the derivation, however, one finds that taking the $m_s \rightarrow \infty$ limit in this way amounts to projecting onto the generators of the appropriate partially quenched SU(2) subgroup. This is exactly analogous to the way in which one projects against the “super- η ” by sending $m_0^2 \rightarrow \infty$ in a PQ calculation [55]. In other words, it is a mathematical trick and does not imply control over the actual m_s dependence.

¹⁸ Here and throughout this Appendix we use m_s to represent both m_s and m_y .

and allows us to piggy-back on the SU(3) SChPT analysis of Ref. [18]. We assume that this all-orders perturbative analysis captures all possible forms of dependence on the small quantities.

The form of the analytic terms in the SU(2) expression, Eqs. (43)–(45), follows immediately in this approach. Indeed, since the SU(2) limit of the NLO SU(3) expression already gives the most general NLO analytic dependence on X_P/Λ^2 , L_P/Λ^2 , $a^2\Lambda_{\text{QCD}}^2$ and α^2 , higher order terms can only generate r_s dependence of their coefficients.

We now turn to the first feature noted above. We start by displaying the form of the contributions proportional to the discretization/truncation LECs that arise in the SU(2) limit of the NLO SU(3) expression. Keeping only the parts which are non-analytic in light-quark quantities, one finds that they all have the form

$$\delta B_K|_{\text{disc/trunc}}^{SU(3)\otimes\text{NLO}} \sim \alpha^2 \frac{M_\pi^2 \log(M_\pi^2)}{G}. \quad (\text{C1})$$

We use here a compact notation: α^2 stands for discretization and truncation effects proportional to α^2 , α/π or $a^2\Lambda_{\text{QCD}}^2$; M_π^2 stands for any light-meson mass-squared [X_B , η_B (except η_1) and L_B] and also (except in the argument of the logarithm) for the hairpin vertices $a^2\delta_B^{\text{MA1,2}}$. In addition, the scale μ_{DR} in the logarithm is kept implicit. The form of (C1) arises as follows: the α^2 comes from the LECs, the $M_\pi^2 \log(M_\pi^2)$ from the pion loop, and the $1/G$ from the denominator in the definition of B_K . All other factors of Λ_{QCD} cancel, as required by dimensional analysis. As noted in Appendix B, the contribution of Eq. (C1) is of NNLO in SU(2) power-counting, because it is suppressed both by α^2 and by M_π^2/G .

The issue then is whether this double suppression continues to hold when one takes the SU(2) limit of higher order SU(3) SChPT expressions. If (C1) simply gets multiplied by a function of r_s , then it remains of NNLO in SU(2) ChPT. If, however, some higher order terms reduce in the SU(2) limit to

$$\delta B_K|_{\text{unwanted}} \sim r_s F(r_s) \alpha^2 \log(M_\pi^2), \quad (\text{C2})$$

then they would contribute at NLO. Here $F(r_s)$ is an unknown function, which we assume to be of $O(1)$ in SU(2) power-counting for $r_s \sim 1$. The overall factor of r_s is included because we know from Eq. (C1) that there is no such term when $r_s = 0$.

That (C2) cannot appear follows from two observations. First, higher orders in SU(3) ChPT cannot introduce an enhancement by $G/M_\pi^2 \propto m_s/m_\ell$. Second, chiral logarithms always have the form $M_\pi^2 \ln(M_\pi^2)$, rather than simply $\ln(M_\pi^2)$. Since the factor of α^2 must be present as we are considering LECs related to discretization or truncation errors, and the $1/G$ arises from the definition of B_K ,¹⁹ one necessarily obtains terms of the form (C1)

rather than (C2).

The two observations of the previous paragraph follow from ChPT power counting, specifically the generalization of staggered ChPT power counting applicable to a calculation of B_K [18]. This power-counting ensures that successive orders in SU(3) SChPT are “suppressed” by factors of m/Λ_{QCD} or α^2 (using the latter generically). Thus it is not possible that by going to higher order one can enhance the result by a factor of m_s/m_ℓ —the best that one can do is have a factor of $r_s \sim O(1)$, which does not change the order in SU(2) ChPT.

As for the pion loop contribution, this can appear as part of an arbitrary-order SU(3) diagram in which all the other particles are “heavy” (i.e. with mass-squared proportional to m_s). In the SU(2) limit this diagram collapses to give a pion tadpole. The contribution of the part of the diagram without the tadpole can be no larger than $\delta B_K \sim \alpha^2 F(r_s)$, where the α^2 comes from the LEC, and SU(3) power-counting ensures that $F(r_s) \sim O(1)$. Adding back in the pion loop, and noting that each pion field comes explicitly with a factor of $1/f$, one sees that the loop must give the generic form

$$\frac{M_\pi^2 \log(M_\pi^2)}{(4\pi f)^2} = \frac{M_\pi^2 \log(M_\pi^2)}{\Lambda_\chi^2} \sim \frac{M_\pi^2 \log(M_\pi^2)}{G},$$

where in the last step we are using the fact that $G \sim \Lambda_\chi$ in SU(2) power counting. In more detail, the loop can involve either a flavor non-singlet pion propagator, in which case $M_\pi^2 = X_B$, or the flavor singlet propagator [e.g. of the form in Eq. (17) for $B = V, A$], in which case the M_π^2 inside the logarithm can be X_B or η_B , while the M_π^2 outside can also be a hairpin vertex proportional to a^2 . In all cases one ends up with an expression of the NNLO form (C1).

We now turn to the second feature of the result Eq. (41) that needs to be explained, namely that the SU(2) chiral logarithm has a predicted coefficient, despite working to all orders in r_s . In the continuum, this follows from an SU(2) ChPT analysis in which the kaon is treated as heavy and serves, along with the four-quark operator, as a source for pions [21]. Using the equations of motion, it is shown in Ref. [21] that all operators contributing to B_K at one-loop order in SU(2) ChPT (which can have any even number of derivatives acting on the external kaons) collapse to a *single* form. Since the coupling to pions is determined by SU(2) chiral symmetry, both the form of the chiral logarithm and its coefficient relative to LO are predicted. The result found in this way is that obtained from Eq. (41) by setting taste-breaking in the pion masses to zero.

Our aim in the following is to show that continuum argument can be generalized to staggered ChPT with only minor changes. Since we have already dealt with the contribution from LECs arising from discretization and truncation errors in the operator, the remaining such errors are those coming from $O(a^2)$ taste-breaking effects

¹⁹ The $1/G$ can be converted into a $1/\Lambda_\chi^2$ by a higher-order correction proportional to $r_s \sim G/\Lambda_\chi^2$. This makes no difference to our arguments since we treat $G \sim \Lambda_\chi^2$ in the SU(2) limit.

arising from the action. These enter through the taste-breaking potential, \mathcal{V} , whose explicit form is given in Refs. [23, 25]. Thus we must consider SU(3) SchPT diagrams with a single insertion of \mathcal{V} and any number of powers of r_s .

The potential \mathcal{V} leads to two-point, four-point and higher-order vertices, all multiplied by an explicit factor of a^2 . Those involving four-point or higher-order vertices give contributions to B_K of the form (C1), which are thus of NNLO in the SU(2) limit. The argument for this is exactly as given above for the discretization/truncation LECs. Thus we need only account for the two-point $O(a^2)$ vertices, which only affect propagators. These will only have an impact on the SU(2) chiral logarithm if these vertices are included in the “light” meson propagators. Including them on “heavy” meson propagators will lead only to corrections of the NNLO form (C1). The upshot is that we should work to all orders in r_s in *continuum* SU(3) ChPT, keeping $O(a^2)$ corrections only on the single light meson propagator needed to develop the non-analyticity. But working to all orders in r_s in SU(3) ChPT can be achieved by using *continuum* SU(2) ChPT—indeed, the latter goes beyond perturbation theory. Thus we end up with the following recipe: use the continuum SU(2) ChPT calculation of Ref. [21] to determine the Feynman diagrams and their overall factors, but evaluate the integrals using the light-meson propagators including the $O(a^2)$ corrections.

In fact, we must generalize the calculation of Ref. [21] to include the extra taste degree of freedom. This is straightforward, and follows the methodology used in Ref. [18]. We find that, just as in SU(3) ChPT, adding taste results in the LO chiral operator having two terms:

$$\mathcal{O}_K^\chi \propto \text{Str}(F_1 \xi K) \text{Str}(F_2 \xi K) + \text{Str}(F_1 \xi K F_2 \xi K) . \quad (\text{C3})$$

Here the pion fields are contained in $\xi = \sqrt{\Sigma} = \exp(i\Phi/[2f])$, while $F_{1,2}$ are spurions which pick out the flavor and taste of the underlying quark operator. The argument for the equality of the coefficients of the two terms is identical to that for SU(3) case given in Ref. [18]. Expanding the operator (C3) to quadratic order in pion fields, removing contributions which are common to the factor of f_K^2 in the denominator of B_K and thus cancel, and contracting pion fields with staggered ChPT propagators leads to Eq. (42).

Appendix D: Tables of am_π and B_K

In this appendix we present the most useful subset of our results for the “pion” masses and B_K . We give results only for the three ensembles, C3, F1 and S1, which we use to do our continuum extrapolation. We also quote

only the degenerate pion masses, since $m_\pi^2 \propto m_x + m_y$ to very good approximation. Finally, for B_K we show the subset of our data used in the SU(2) fits. These are the results that lead our central value for B_K . Table XXXI reports the average pion masses obtained as described in the main text. Table XXXII reports the values of B_K obtained using 1-loop matching at the scale $\mu = 1/a$.

TABLE XXXI. am_π (Goldstone taste, ξ_5) for the C3, F1 and S1 ensembles.

quark comb.	C3	F1	S1
1-1	0.1342(2)	0.0902(3)	0.0596(2)
2-2	0.1861(2)	0.1244(2)	0.0819(2)
3-3	0.2258(2)	0.1507(2)	0.0992(1)
4-4	0.2593(2)	0.1728(2)	0.1139(1)
5-5	0.2888(2)	0.1924(2)	0.1269(1)
6-6	0.3155(2)	0.2101(2)	0.1387(1)
7-7	0.3402(2)	0.2266(2)	0.1496(1)
8-8	0.3634(2)	0.2419(2)	0.1599(1)
9-9	0.3852(2)	0.2565(2)	0.1695(1)
10-10	0.4060(2)	0.2703(2)	0.1788(1)

TABLE XXXII. Results for $B_K(1/a)$ on the C3, F1, and S1 ensembles.

quark comb.	C3	F1	S1
1-1	0.438(12)	0.360(33)	0.357(23)
2-2	0.4815(46)	0.434(14)	0.4147(73)
3-3	0.5132(27)	0.4699(77)	0.4455(41)
4-4	0.5363(19)	0.4928(53)	0.4689(27)
5-5	0.5547(15)	0.5118(40)	0.4878(21)
6-6	0.5701(12)	0.5283(33)	0.5036(17)
7-7	0.5835(10)	0.5426(27)	0.5173(14)
8-8	0.5954(9)	0.5553(23)	0.5294(13)
9-9	0.6061(8)	0.5667(20)	0.5403(12)
10-10	0.6159(7)	0.5771(18)	0.5502(11)
1-8	0.5735(38)	0.5243(97)	0.4908(58)
2-8	0.5674(21)	0.5215(55)	0.4915(30)
3-8	0.5690(16)	0.5248(42)	0.4978(22)
4-8	0.5732(13)	0.5303(35)	0.5045(18)
5-8	0.5784(11)	0.5365(31)	0.5111(16)
1-9	0.5811(38)	0.5329(95)	0.4992(57)
2-9	0.5759(20)	0.5304(53)	0.5000(29)
3-9	0.5772(15)	0.5332(40)	0.5058(21)
4-9	0.5809(12)	0.5382(33)	0.5119(17)
5-9	0.5855(11)	0.5439(29)	0.5180(15)
1-10	0.5879(38)	0.5410(93)	0.5070(56)
2-10	0.5835(20)	0.5386(51)	0.5078(29)
3-10	0.5848(14)	0.5410(38)	0.5132(20)
4-10	0.5881(12)	0.5455(31)	0.5188(17)
5-10	0.5922(10)	0.5507(27)	0.5245(15)

[1] C. Amsler and *et al.* (Particle Data Group), <http://pdg.lbl.gov/> (2009).

[2] A. J. Buras and D. Guadagnoli, Phys. Rev. D **78**, 033005 (2008).

- [3] C. Aubin, J. Laiho, and R. S. V. de Water, Phys. Rev. **D81**, 014507 (2010), arXiv:0905.3947.
- [4] C. Allton et al. (RBC-UKQCD), Phys. Rev. **D78**, 114509 (2008), arXiv:0804.0473.
- [5] C. Kelly, P. A. Boyle, and C. T. Sachrajda (RBC), PoS **LAT2009**, 087 (2009), arXiv:0911.1309.
- [6] W. Lee and S. R. Sharpe, Phys. Rev. **D66**, 114501 (2002), hep-lat/0208018.
- [7] W. Lee, PoS **LAT2006**, 015 (2006), arXiv:hep-lat/0610058.
- [8] E. Gamiz, S. Collins, C. Davies, G. Lepage, J. Shigemitsu, and M. Wingate (HPQCD), Phys. Rev. **D73**, 114502 (2006), arXiv:hep-lat/0603023.
- [9] C. Bernard, Phys. Rev. **D73**, 114503 (2006), hep-lat/0603011.
- [10] S. R. Sharpe, PoS **LAT2006**, 022 (2006), hep-lat/0610094.
- [11] C. Bernard, M. Golterman, and Y. Shamir, Phys. Rev. **D77**, 074505 (2008), arXiv:0712.2560.
- [12] Y. Shamir, Phys. Rev. **D75**, 054503 (2007), hep-lat/0607007.
- [13] A. Bazavov et al., Rev. Mod. Phys. **82**, 1349 (2010), arXiv:0903.3598.
- [14] H.-J. Kim, T. Bae, J. Kim, J. Kim, K. Kim, B. Yoon, W. Lee, C. Jung, and S. R. Sharpe, PoS (**Lattice 2009**), 262 (2009), arXiv:0910.5573.
- [15] T. Bae, H.-J. Kim, J. Kim, J. Kim, K. Kim, B. Yoon, W. Lee, C. Jung, and S. R. Sharpe, PoS (**Lattice 2009**), 261 (2009), arXiv:0910.5576.
- [16] B. Yoon, T. Bae, H.-J. Kim, J. Kim, J. Kim, K. Kim, W. Lee, C. Jung, and S. R. Sharpe, PoS (**Lattice 2009**), 263 (2009), arXiv:0910.5581.
- [17] J. Kim, T. Bae, H.-J. Kim, J. Kim, K. Kim, B. Yoon, W. Lee, C. Jung, and S. R. Sharpe, PoS (**Lattice 2009**), 264 (2009), arXiv:0910.5583.
- [18] R. V. de Water and S. Sharpe, Phys. Rev. **D73**, 014003 (2006), arXiv:hep-lat/0507012.
- [19] C. Aubin and *et al.* (MILC Collaboration), Phys. Rev. **D70**, 114501 (2004), arXiv:hep-lat/0407028.
- [20] C. Allton et al. (RBC-UKQCD), Phys. Rev. **D78**, 114509 (2008), arXiv:0804.0473.
- [21] D. J. Antonio et al. (RBC), Phys. Rev. Lett. **100**, 032001 (2008), hep-ph/0702042.
- [22] W. Lee and S. R. Sharpe, Phys. Rev. **D68**, 054510 (2003), hep-lat/0306016.
- [23] W. Lee and S. R. Sharpe, Phys. Rev. **D60**, 114503 (1999), hep-lat/9905023.
- [24] C. Bernard (MILC Collaboration), private communication (2009).
- [25] C. Aubin and C. Bernard, Phys. Rev. **D68**, 034014 (2003), arXiv:hep-lat/0304014.
- [26] O. Bar, G. Rupak, and N. Shores, Phys. Rev. **D67**, 114505 (2003), hep-lat/0210050.
- [27] O. Bar, C. Bernard, G. Rupak, and N. Shores, Phys. Rev. **D72**, 054502 (2005), hep-lat/0503009.
- [28] M. Golterman, T. Izubuchi, and Y. Shamir, Phys. Rev. **D71**, 114508 (2005), hep-lat/0504013.
- [29] J.-W. Chen, M. Golterman, D. O'Connell, and A. Walker-Loud, Phys. Rev. **D79**, 117502 (2009), 0905.2566.
- [30] T. Bae, D. H. Adams, C. Jung, H.-J. Kim, J. Kim, K. Kim, W. Lee, and S. R. Sharpe, Phys. Rev. **D77**, 094508 (2008), arXiv:0801.3000.
- [31] J. Kim, W. Lee, and S. R. Sharpe, PoS **LAT2009**, 201 (2009), arXiv:0910.5586.
- [32] Y. Kuramashi, PoS **LATTICE2008**, 018 (2008), arXiv:0811.2630.
- [33] L. Lellouch, PoS **LATTICE 2008**, 015 (2008), arXiv:0902.4545.
- [34] P. A. Boyle (2009), arXiv:0911.4317.
- [35] J. F. Donoghue (2009), arXiv:0909.0021.
- [36] A. Roessl, Nucl. Phys. **B555**, 507 (1999), hep-ph/9904230.
- [37] A. Hasenfratz and F. Knechtli, Phys. Rev. **D64**, 034504 (2001), hep-lat/0103029.
- [38] C. T. H. Davies, E. Follana, I. D. Kendall, G. P. Lepage, and C. McNeile (HPQCD) (2009), arXiv:0910.1229.
- [39] G. W. Kilcup, S. R. Sharpe, R. Gupta, and A. Patel, Phys. Rev. Lett. **64**, 25 (1990).
- [40] G. Kilcup, R. Gupta, and S. R. Sharpe, Phys. Rev. **D57**, 1654 (1998), hep-lat/9707006.
- [41] W. Lee, T. Bhattacharya, G. T. Fleming, G. Kilcup, R. Gupta, and S. R. Sharpe, Phys. Rev. **D71**, 094501 (2005), hep-lat/0409047.
- [42] D. Adams, T. Bae, H.-J. Kim, J. Kim, K. Kim, B. Yoon, W. Lee, C. Jung, and S. R. Sharpe, PoS **LATTICE 2008**, 104 (2008), arXiv:0809.1219.
- [43] J. Kim, W. Lee, and S. Sharpe, in preparation (2010).
- [44] A. J. Buras, M. Jamin, and P. Weisz, Nucl. Phys. **B347**, 491 (1990).
- [45] R. Gupta, T. Bhattacharya, and S. R. Sharpe, Phys. Rev. **D55**, 4036 (1997), hep-lat/9611023.
- [46] G. Lepage, B. Clark, C. Davies, K. Hornbostel, P. Mackenzie, C. Morningstar, and H. Trottier, Nucl. Phys. (Proc. Suppl.) **B106**, 12 (2002), arXiv:hep-lat/0110175.
- [47] C. Jung, PoS **LAT2009**, 2 (2010), arXiv:1001.0941.
- [48] K. Nakamura et al. (Particle Data Group), J. Phys. **G37**, 075021 (2010).
- [49] M. Constantinou et al. (ETM) (2010), 1009.5606.
- [50] S. Aoki et al. (JLQCD), Phys. Rev. **D77**, 094503 (2008), 0801.4186.
- [51] Y. Aoki et al., Phys. Rev. **D72**, 114505 (2005), hep-lat/0411006.
- [52] J. M. Flynn, F. Mescia, and A. S. B. Tariq (UKQCD), JHEP **11**, 049 (2004), hep-lat/0406013.
- [53] A. T. Lytle, PoS **LAT2009**, 202 (2009), 0910.3721.
- [54] S. R. Sharpe, Phys. Rev. **D46**, 3146 (1992), arXiv:hep-lat/9205020.
- [55] S. R. Sharpe and N. Shores, Phys. Rev. **D64**, 114510 (2001), hep-lat/0108003.